Nominal Bonds and Interest Rates: 
Liquidity and Welfare*

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Abstract

In this paper I analyze how interest rates, output and welfare depend on the liquidity of nominal bonds. The model features limited participation in the bonds market and decentralized exchanges (search) in the goods market. In a fraction of trades in the goods market, a legal restriction forbids the use of bonds as the means of payments. I show that nominal bonds can become illiquid endogenously when tastes are strong for the goods in the restricted trades. When such tastes are not very strong, bonds are liquid and the yield curve is negatively sloped, but bonds may or may not be perfect substitutes for money. As the coverage of the legal restriction increases, welfare first decreases and then increases, but eliminating the legal restriction altogether increases welfare. I also compare economies with exogenous degrees of bonds’ liquidity. Completely illiquid bonds can yield higher welfare and higher nominal interest rates than liquid bonds. However, between two economies in which bonds have interior degrees of liquidity, it is possible that welfare is higher when bonds are more liquid, that welfare is higher when the nominal interest rate is lower, and that the interest rate is higher when bonds are more liquid.

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1. Introduction

The competition between money and nominal bonds is a classical issue in monetary economics (e.g. Hicks 1939). When nominal bonds are default-free, they have all the essential features that money has. It is then puzzling to observe that nominal bonds do not act as a medium of exchange to the same extent as money does and that they are discounted. This puzzle contains two distinct questions. First, why do not matured bonds circulate perpetually as money does? Second, why do not unmatured bonds circulate as widely as money? As Wallace (2001) argued, these issues can be best addressed in a search monetary model, which has a strong microfoundation.

In a precursor (Shi, 2002) to the current paper, I answered the first question by introducing into a search monetary model a legal restriction, which forbids the use of bonds as the means of payments for goods in a fraction of trades. This legal restriction drives matured nominal bonds out of the goods market even when the restriction covers only an arbitrarily small fraction of trades. In that analysis, nominal bonds are one-period bonds, although the conclusion applies generally to all matured bonds. In the current paper, I address the second question by introducing long-term bonds so that unmatured bonds have a chance to circulate in the goods market. I analyze how nominal interest rates, output and welfare depend on the liquidity of nominal bonds and the coverage of the legal restriction.

The basic model is a hybrid of the deterministic version of Lucas’s (1990) model of limited participation and the search model of money in Shi (1997, 1999). The bonds market is centralized (Walrasian). The goods market is decentralized, with random matching. By a legal restriction, agents cannot use bonds to buy goods in a fraction \( g \) of trades, which are called restricted trades. Other trades are unrestricted, in which both money and bonds can be used. The two markets are separated from each other and a household must choose the division of the assets between the two markets before the markets open. Liquidity of bonds refers to the endogenous fraction of unmatured bonds taken to the goods market.

The equilibrium depends on a parameter \( \theta \), which measures how much an agent values the goods in restricted trades relative to the goods in unrestricted trades. When \( \theta \) is small, unmatured bonds are fully liquid and they are perfect substitutes for money; When \( \theta \) has intermediate values, bonds are perfectly liquid but they are imperfect substitutes for money; When \( \theta \) is large, unmatured bonds are poor substitutes for money, and the equilibrium is consistent with a range of values of bonds’ liquidity. Therefore, perfect illiquidity of bonds is an endogenous outcome when tastes are strong enough for the goods in restricted trades. For this critical level of \( \theta \) to be finite, the coverage \( (g) \) of the legal restriction must be bounded above 0.

The (two-period) bond price is higher, and hence the nominal interest rate lower, when unmatured bonds are better substitutes for money in the goods market. Thus, the yield curve is more negatively sloped when bonds are better substitutes for money. Moreover, a two-period bond in
its second period can have a higher price than a newly issued one-period bond.

In this economy, the legal restriction affects welfare in a U-shaped pattern. The legal restriction creates a gap between the quantities of goods traded in a restricted match and an unrestricted match. This gap is irrelevant either when the legal restriction does not exist or when the restriction covers all trades. Welfare is the highest in these two polar cases. When the legal restriction covers only a fraction of trades, an increase in the coverage reduces the gap between the quantities of goods in the two types of trades, which improves welfare by smoothing consumption between the trades. Also, an increase in the coverage shifts the composition of trades from unrestricted ones to restricted ones, which reduces welfare. The first effect dominates when the coverage of the restriction is already wide. In this case, a further increase in the coverage improves welfare, although eliminating the restriction altogether always improves welfare. The opposite result occurs when the coverage of the legal restriction is small.

Real activities and welfare also depend on the liquidity of nominal bonds. To examine this dependence, I analyze economies where bonds’ liquidity is exogenously fixed. An increase in bonds’ liquidity redistributes consumption from restricted trades to unrestricted trades, by raising the price level and increasing the amount of assets that can be used in an unrestricted trade. When tastes are nearly symmetric over the goods, this redistribution reduces welfare by making consumption less smooth between the two types of trades. In this case, making bonds completely illiquid improves welfare and increases the nominal interest rate.

However, if the liquidity of bonds is interior, the interest rate may not be monotonically associated with bonds’ illiquidity. The interest rate decreases with bonds’ liquidity for small $\theta$, but is independent of bonds’ liquidity for large $\theta$. For $\theta$ near 1 (which is the case where completely illiquid bonds can improve welfare over liquid bonds), the interest rate first decreases and then increases with bonds’ liquidity.

Similarly, there is no systematic relationship between the welfare level and the interest rate when the liquidity of bonds is interior. For example, when $\theta$ is near 1, welfare decreases monotonically with the degree of bonds’ liquidity, but the dependence of the interest rate on bonds’ liquidity is U-shaped. In this case, the interest rate and welfare are positively related to each other when bonds’ liquidity is low, but negatively related when bonds’ liquidity is high.

This paper is closely related to Aiyagari et al. (1996), who examine nominal bonds in a search monetary model. They impose two restrictions that are absent in my model. One is that money and bonds are indivisible. The other is that the bonds market is characterized by random matching, and so the purchase and redemption of nominal bonds are stochastic events. By assuming a centralized bonds market and making the assets divisible, I allow for greater competition between nominal bonds and money, and hence generate results that are more robust. Also, the model is a convenient vehicle for analyzing standard issues in monetary economics, such as money growth, open market operations, and interest rates.
Another closely related paper is Kocherlakota (2001), who introduces taste shocks into a model of spatially separated markets. He shows that completely illiquid bonds improve welfare over liquid bonds and that the nominal interest rate is higher with illiquid bonds. My model shares these results, but differs substantially in other aspects. First, I allow the legal restriction to cover only a fraction of the trades, while Kocherlakota assumes that nominal bonds are either completely liquid or illiquid. The partial coverage enables me to establish the novel results that nominal bonds can become illiquid endogenously and that welfare depends on the coverage of the legal restriction in a U-shaped pattern. It also enables me to disentangle the liquidity of bonds from the perfect substitutability of bonds for money. Second, I examine the entire spectrum of bonds’ liquidity, rather than the two extremes. This examination reveals that, when bonds’ liquidity is interior, welfare is not monotonically related to the nominal interest rate or to the degree of bonds’ liquidity. Finally, I examine the relationship between the liquidity and maturity of nominal bonds, which Kocherlakota does not do.¹

2. A Search Economy with Nominal Bonds

In this section I describe an economy without the legal restriction in the goods market and characterize the equilibrium. This equilibrium illustrates that unmatured bonds will circulate in the goods market at par with money.

2.1. Households, Matches, and Timing

Consider a discrete-time economy with many types of households. The number of households in each type is large and normalized to one. The households in each type are specialized in producing a specific good, which they do not consume, and exchange for consumption goods in the market. Goods are perishable between periods. The utility of consumption is \( u(.) \) for consumption goods and 0 otherwise. The cost (disutility) of production is \( \psi(.) \). The utility function satisfies \( u'(0) > 0 \) and \( u'' \leq 0 \). The cost function satisfies \( \psi(0) = 0, \psi'(0) > 0 \) and \( \psi'' > 0 \). Moreover, \( u'(0) = \infty > \psi'(0) \) and \( u'(\infty) = 0 < \psi'(\infty) \).

Agents meet their trading partners bilaterally and randomly in the market. There is no chance for a double coincidence of wants in a meeting to support barter or public record-keeping of transactions to support credit trades. As a result, every trade entails a medium of exchange.

Two objects compete to serve as the medium of exchange. One is money and the other is nominal bonds issued by the government. These objects (assets) can be stored without cost. Both are intrinsically worthless; i.e., they do not yield direct utility or facilitate production. To

¹After I wrote the precursor to the current paper (Shi, 2002) and did the preliminary work on the current paper, I became aware of a paper by Rocheteau (2002), who uses a search monetary model to examine the same legal restriction in the goods market as in this paper. However, he does not examine how bonds’ liquidity depends on the maturity, or how welfare and nominal interest rates vary with the degree of bonds’ liquidity. A further comparison with Rocheteau’s model will appear in section 4.
eliminate risks as the reason for discounting on bonds, I assume that the bonds are default-free. Each bond can be redeemed for one unit of money at (and only at) maturity. Also, I set the length of maturity of bonds to be two periods – the shortest length that gives bonds a chance to circulate in the goods market before maturity. The bonds one period after the issuing date are called *unmatured bonds*.

To specify the matching technology, I call an agent in the goods market a buyer if he holds money or bonds, and a seller if he holds neither money nor bonds. A seller produces and sells goods, and a buyer purchases consumption goods. Let $\sigma$ be the (fixed) fraction of sellers and $(1 - \sigma)$ the fraction of buyers in the market. Of interest are the meetings of a single coincidence of wants, i.e., meetings in which one and only one agent can produce the partner’s consumption goods. Call such a meeting a *trade match*. A buyer encounters a trade match in a period with probability $\alpha \sigma$ and a seller with probability $\alpha (1 - \sigma)$, where $\alpha > 0$ is a constant.

Random matching generates non-degenerate distributions of agents’ money holdings and consumption. To maintain tractability, I assume that each household consists of a large number of members (normalized to one) who share consumption each period and regard the household’s utility function as the common objective. This assumption makes the distribution of money holdings across households degenerate and hence allows me to focus on equilibria that are symmetric across households.\(^2\) In each household, there are a measure $\sigma$ of sellers and $(1 - \sigma)$ of buyers. A buyer can combine money and bonds to purchase goods in a trade.\(^3\)

The bonds market is centralized and has a much lower transaction cost than in the goods market. To simplify, I abstract from such a transaction cost altogether and, in particular, assume that trades in the bonds market take zero measure of members. As in Lucas (1990), the government sells bonds only for money. The market price of newly issued (two-period) bonds is $S$ and redemption of a bond at maturity yields one unit of money. The two-period (net) nominal interest rate is $r = 1/S - 1$. Let $S^u$ be the nominal price of unmatured bonds in the bonds market. Then, $1/S^u$ is the one-period nominal interest rate implied by the bond of the old vintage.

To describe the timing of events in a period, pick an arbitrary period $t$, suppress the time index $t$, and shorten the time subscript $t \pm j$ as $\pm j$ for $j \geq 1$. Also, pick an arbitrary household as the representative household. Lower-case letters denote the decisions of this household and capital-case letters other households’ decisions or aggregate variables. I depict the timing of events in a period in Figure 1.

At the beginning of the period the household redeems bonds that were issued two periods ago and receives a lump-sum monetary transfer. After these events, the household’s holding of

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\(^2\)The assumption of large households is a modelling device extended from Lucas (1990). Lagos and Wright (2001) use a different set of assumptions to achieve essentially the same purpose of risk smoothing.

\(^3\)This is not a critical assumption. Alternatively, one can assume that each buyer can carry either money or bonds, but not both, into a match. I examined this alternative specification with one-period bonds (Shi, 2002) and found that the qualitative results are the same.
money is denoted \( m \) and of unmatured bonds \( b \). The monetary transfer keeps the money holding per household growing at a constant rate \( \gamma \); that is, \( M = \gamma M_{-1} \).

The household divides the assets into two parts. A fraction \( a \) of money and a fraction \( l \) of unmatured bonds are allocated to the goods market, where \( l \) indicates “liquid” bonds, while the remaining assets are allocated to the bonds market. The household divides the assets for the goods market evenly among the buyers. Each buyer carries \( am/(1 - \sigma) \) units of money and \( lb/(1 - \sigma) \) units of unmatured bonds into the goods market. As stated earlier, a buyer can use both money and unmatured bonds to exchange for goods in a trade.

\[
\begin{align*}
\text{redemption} & \quad \text{portfolio} \quad (a, l) \quad \text{markets} \quad \text{open} \quad \text{markets} \quad \text{closed} \quad t + 1 \\
\text{money transfers} & \quad (m, b) \quad \text{decisions on} \quad \text{trades: } d, b^u \quad \text{pooling,} \quad \text{consumption} \\
& \quad \text{measured} \quad \text{goods trade} \quad (q, x) \quad \text{and } (q, x) \quad \text{consume}
\end{align*}
\]

Figure 1 Timing of events in a period

At the time of choosing the portfolio divisions \((a, l)\), the household also chooses the quantities of trade \((q, x)\), which I will describe later.

Next, the two markets open simultaneously and separately. In the bonds market, the household uses money to purchase new bonds and can also sell or buy unmatured bonds. Let \( d \) be the amount of new bonds that the household purchases and \( b^u \) the amount of unmatured bonds with which the household exits the bonds market. In the goods market, the agents trade according to the quantities \((q, x)\) prescribed by the household. Then, the markets close. The household pools the receipts from the trades and allocates consumption evenly among members. After consumption, time proceeds to the next period.

As in Lucas (1990), the temporary separation between the two markets implies an opportunity cost for bringing assets into the bonds market, and hence a discount on new bonds. In contrast to Lucas’s model, the goods market is decentralized here, rather than centralized. Also, households can use unmatured bonds, as well as money, to buy goods. Lucas assumes that money is the only medium of exchange, which imposes \( l = 0 \) a priori.

2.2. Quantities of Trade in the Goods Market

The household chooses the quantities of money and goods to be traded in each trade match. To describe these choices, let \( \beta \in (0, 1) \) be the discount factor, and \( v(m, b) \) the household’s value function after redeeming matured bonds and receiving the transfer in the period. Let \( \omega^i \) be the shadow value of next period’s asset \( i \) \((= m, b)\), discounted to the current period. That is,

\[
\omega^m = \beta v_1(m_{+1}, b_{+1}), \quad \omega^b = \beta v_2(m_{+1}, b_{+1}),
\]

(2.1)
where the subscripts of $v$ indicate partial derivatives and the subscript +1 indicates “the next period”. Other households’ values of the two assets are $\Omega^m$ and $\Omega^b$, respectively.

To simplify the analysis, I assume that the buyer in a trade makes a take-it-or-leave-it offer. The offer specifies the quantity of goods that the buyer asks the seller to supply, $q$, and the quantity of assets that the buyer gives, $x$. Notice that it is unnecessary to specify the division of the amount $x$ between money and unmatured bonds, because the two assets are equivalent to the receiver (seller): Upon exiting from the trade, the receiver will not have the opportunity to use the assets to purchase goods in the current period and, at the beginning of next period, the received bonds mature and can be redeemed for money at par.\(^4\)

When choosing the quantities $(q, x)$, the household anticipates the following constraints:

\[
x \leq \frac{am + lb}{1 - \sigma}, \quad (2.2)
\]
\[
\psi(q) \leq \Omega^m x. \quad (2.3)
\]

The first constraint says that the buyer cannot offer more assets than what he brought into the trade. The second constraint says that the seller must be willing to accept the offer. I will call (2.2) the asset constraint in the goods market. When this constraint binds, I say that assets yield liquidity service in the goods market.

### 2.3. A Household’s Decision Problem

The household’s choices in each period are the portfolio divisions, $(a, l)$, the quantities of trade, $(q, x)$, the amount of new bonds to purchase, $d$, the amount of unmatured bonds to hold exiting the bonds market, $b^u$, consumption, $c$, and future asset holdings, $(m_{+1}, b_{+1})$. Taking other households’ choices and aggregate variables as given, the choices solve the following problem:

\[
(PH) \quad v(m, b) = \max \{u(c) - \alpha \sigma (1 - \sigma) \psi(Q) + \beta v(m_{+1}, b_{+1})\}.
\]

The constraints are as follows:

(i) the constraints in the goods market, (2.2) and (2.3), and $c = \alpha \sigma (1 - \sigma) q$;

(ii) the constraints in the bonds market: $b^u \geq 0$ and

\[
Sd \leq (1 - a)m + S^u [(1 - l)b - b^u]; \quad (2.4)
\]

(iii) the laws of motion of asset holdings:

\[
b_{+1} = d, \quad (2.5)
\]
\[
m_{+1} = m - Sd + S^u [(1 - l)b - b^u] + \alpha \sigma (1 - \sigma) (X - x) + (lb + b^u) + L_{+1}; \quad (2.6)
\]

\(^4\)For the same reason, a trade in the goods market between a money holder and a bond holder is inconsequential, and so it is omitted here.
(iv) and other constraints: $0 \leq a \leq 1$ and $0 \leq l \leq 1$.

Consumption is equal to the amount of goods obtained by the buyers in the period, where the total number of such trades is $\alpha \sigma (1 - \sigma)$. The disutility of production is computed similarly, but with $Q$ replacing $q$, because other households’ buyers choose the quantity of goods to be traded in such matches. The constraints in (i) and (iv) are self-explanatory.

The constraint $b^u \geq 0$ in (ii) requires that the household should not sell more unmatured (government) bonds than the amount it brought into the bonds market.\(^5\) The constraint (2.4) states that money spent on newly issued bonds comes from money brought into the bonds market plus the receipt from selling unmatured bonds.

The law of motion of bonds, (2.5), is straightforward – newly issued bonds in this period become unmatured bonds next period. To explain the law of motion of money, (2.6), recall that the household’s money holding is measured at the time immediately after receiving monetary transfers and redeeming matured bonds (see Figure 1). This holding can change between two adjacent periods as a result of the following transactions: purchasing newly issued bonds, selling unmatured bonds in the bonds market, selling and buying goods, redeeming matured bonds and receiving the monetary transfer next period. The terms following $m$ on the right-hand side of (2.6) list the net changes in money holdings from these five types of transactions.

To characterize optimal decisions, let $\rho$ be the Lagrangian multiplier of the constraint in the bonds market, (2.4), and $\lambda$ of the asset constraint in the goods trade, (2.2). To simplify the equations, multiply $\lambda$ by the number of trades that involve the household’s buyers, $\alpha \sigma (1 - \sigma)$. Incorporating these constraints, I can modify the objective function in (PH) as follows:

$$v(m, b) = \max \{ \beta v(m+1, b+1) + u(\alpha \sigma (1 - \sigma)q) - \alpha \sigma (1 - \sigma)\psi(Q)$$
$$+ \rho [(1 - a) m + S^u((1 - l)b - b^u) - Sd] + \alpha \sigma (1 - \sigma)\lambda \left( \frac{am + \lambda b^u}{1 - \sigma} - x \right) \}.$$  

The quantity $x$ satisfies (2.3) with equality (provided $\omega^m > 0$), and so

$$x = \psi(q)/\Omega^m.$$  

The following conditions are necessary for the decisions to be optimal:

(i) For $q$:

$$u'(c) = (\omega^m + \lambda) \psi'(q) / \Omega^m.$$  

(ii) For $(a, d, l, b^u)$:

$$\alpha \sigma \lambda = \rho \quad \text{if} \ a \in (0, 1),$$

$$\omega^b = (\omega^m + \rho) S \quad \text{if} \ d \in (0, \infty),$$

$$\alpha \sigma \lambda + \omega^m = (\omega^m + \rho) S^u \quad \text{if} \ l \in (0, 1),$$

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\(^5\)This constraint arises because the government does not buy back unmatured bonds. Individual households can issue private bonds but, because all households are symmetric, such private bonds do not affect the equilibrium.
\[ \omega^m = (\omega^m + \rho)S^u \quad \text{if } b^u > 0. \quad (2.12) \]

In each of these conditions, the variable attains the lowest value in the specified domain if the equality is replaced by “<”, and the highest value if “>”.

(iii) For \((m, b)\) (envelope conditions):

\[ \frac{\omega^m_1}{\beta} = \omega^m + (1 - a)\rho + a\alpha\sigma\lambda, \quad (2.13) \]

\[ \frac{\omega^b_1}{\beta} = (1 - l)(\omega^m + \rho)S^u + l(\omega^m + \alpha\sigma\lambda). \quad (2.14) \]

The condition (2.8) requires that a buyer’s net gain from asking for an additional amount of goods be zero. By getting an additional unit of good, the household’s utility increases by \(u'(c)\). The cost is to pay an additional amount \(\psi'(q)/\Omega^m\) of assets in order to induce the seller to trade (see (2.7)). By giving an additional unit of asset, the buyer foregoes the discounted future value of the asset, \(\omega^m\), and causes the asset constraint in the trade to be more binding. Thus, \((\omega^m + \lambda)\) is the shadow cost of each additional unit of asset to the buyer’s household and the right-hand side of (2.8) is the cost of getting an additional unit of good from the seller.

In (ii), (2.9) says that for the household to allocate money to both the goods market and the bonds market, money must generate the same marginal liquidity service in the two markets by relaxing the money constraints. For the amount of newly issued bonds, (2.10) requires the expected future value of such bonds to be equal to the cost of money that is used to acquire them, including the shadow cost of the money constraint in the bonds market (\(\rho\)).

For unmatured bonds, the condition (2.11) characterizes the decision for \(l\) by comparing the shadow values of unmatured bonds in the two markets. The shadow value of an unmatured bond in the goods market is \((\alpha\sigma\lambda + \omega^m)\), because the bond can relieve the asset constraint and be redeemed for one unit of money in the future. The shadow value of an unmatured bond in the bonds market is \((\omega^m + \rho)S^u\), because the bond can be sold for \(S^u\) units of money and each unit of money has a shadow value \((\omega^m + \rho)\) in the bonds market. If the optimal choice of \(l\) is interior, then these two shadow values must be equal to each other. The condition (2.12) characterizes the decision for \(b^u\), the amount of unmatured bonds that the household exits the bonds market with. This amount is positive if and only if the shadow value of keeping such a bond and redeeming it next period is equal to the shadow value of selling it now for money.

Finally, the envelope conditions require the current value of each asset to be equal to the sum of the future value of the asset and the expected liquidity service generated by the asset in the current markets. Take money for example. The current value of money is given by the left-hand side of (2.13), where \(\omega^m_1\) is divided by \(\beta\) because \(\omega^m_1\) is defined as the value of money discounted to one period earlier. The right-hand side of (2.13) consists of the (discounted) future value of money, \(\omega^m\), the liquidity service generated by money in the current bond market, \(\rho\), and the liquidity service generated by money in the current goods market, \(\alpha\sigma\lambda\). The services in the two markets are weighted by the division of money into the two markets.
2.4. Symmetric Equilibrium

A symmetric monetary equilibrium consists of a sequence of a representative household’s choices 

\[(a_t, l_t, q_t, x_t, d_t, b_t^m, c_t, m_{t+1}, b_{t+1})_{t=0}^{\infty},\]

the implied shadow prices \((\omega_t^m, \omega_t^b, \rho_t, \lambda_t)_{t=0}^{\infty}\) and other households’ choices (capital-case variables) such that the following requirements are met. (i) Optimality: given other households’ choices, the household’s choices solve \((PH)\) with given initial holdings \((m_0, b_0)\); (ii) Symmetry: the choices (and shadow prices) are the same across households; (iii) Clearing for newly issued bonds: \(d_t = zM_t\), with \(z > 0\); (iv) Clearing for unmatured bonds: \(b_t^u = (1 - L_t)B_t\); (v) Positive and finite values of assets: \(0 < \omega_t^m m_t / \beta < \infty\) and \(0 < \omega_t^b b_t / \beta < \infty\) for all \(t\) if \(m, b > 0\); (vi) Stationarity: all real variables and the values \((\omega_{t-1}^m m, \omega_{t-1}^b b)\) are constant.

In this definition I have restricted the amount of newly issued bonds to be a constant fraction of the money stock. The total value of each asset is restricted to be positive and finite, in order to examine the coexistence of money and bonds.\(^6\) Furthermore, I restrict attention to equilibria in which money serves as a medium of exchange. This restriction imposes two requirements. First, the money growth rate must satisfy \(\gamma > \beta\), because money serves only as a store of value for \(\gamma = \beta\) and a monetary equilibrium does not exist for \(\gamma < \beta\). Second, \(a > 0\). Since the market clearing condition for newly issued bonds implies \(a < 1\), then \(0 < a < 1\), and so \(\rho = \alpha \sigma \lambda \beta^{-(1)}\) by (2.9).

To characterize the symmetric equilibrium, define a function \(f\) and a constant \(\mu\) as follows:

\[\frac{u'(\alpha \sigma (1 - \sigma) f(k))}{\psi''(f(k))} = k, \text{ for } k > 0,\]

(2.15)

\[\mu = \frac{1}{\alpha \sigma} \left(\frac{\gamma}{\beta} - 1\right).\]

(2.16)

The function \(f\) is a decreasing function. I prove the following proposition in Appendix A:

**Proposition 2.1.** Assume that the size of the bond sales satisfies \(z \in (0, \gamma / \beta)\). When there is no legal restriction in the goods market, an equilibrium exists and it is unique for all real variables. In this equilibrium, \(\omega^b = \omega^m, l = 1\), and the price of newly issued (two-period) bonds is:

\[S = \beta / \gamma \quad (< 1).\]

(2.17)

The price of the unmatured bonds (in the bonds market) is indeterminate: \(\beta / \gamma \leq S^u \leq 1\). The quantity of goods traded in a match is \(q = f(1 + \mu)\) and the fraction of money taken to the goods market is \(a = 1 - z \beta / \gamma\).

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\(^6\)The bounded value of each asset is necessary for the household’s optimal decisions to be indeed characterized by the first-order conditions obtained in the previous section.
Unmatured bonds are perfect substitutes for money in the goods market (i.e., $ω^b = ω^m$). This result holds because, as explained before, the seller in a match is indifferent between receiving money and unmatured bonds as payments for his goods; both assets must be carried to the next period at which time they are equivalent to each other.

Households take all unmatured bonds to the goods market; i.e., bonds are perfectly liquid. To see why this occurs in the equilibrium, suppose to the contrary that a household brings some unmatured bonds to the bonds market. For this decision to be optimal, unmatured bonds must be sold at par for money in the bonds market, because they are perfect substitutes for money in the goods market. Because the money constraint on the purchase of newly issued bonds binds when $γ > β$, zero discounting on unmatured bonds implies that all households will sell unmatured bonds for money. There is an excess supply of unmatured bonds in the bonds market, which contradicts the equilibrium.

Newly issued bonds are sold at a discount, resulting in a positive net nominal interest rate. The reason is that the separation between the two markets prevents a household from using new bonds to purchase goods immediately. When allocating money to purchase the new bonds, a household incurs a one-period loss of liquidity which money can generate in the goods market. To compensate for this one-period loss of liquidity, a newly issued bond must be discounted, although this bond will be equivalent to a unit of money next period. In fact, the one-period loss of liquidity is the common cause for discounting on all new bonds. That is, all newly issued bonds have the same price independently of their lengths of maturity, and so the yield curve of the term structure is negatively sloped.

The (resale) price of unmatured bonds is indeterminate in the bonds market, because unmatured bonds do not appear in the bonds market. The demand for unmatured bonds in the bonds market is zero when $S^u > β/γ$ and the supply is zero if $S^u < 1$. Including the two borderline cases, all values $S^u ∈ [β/γ, 1]$ are consistent with equilibrium.

If one-period bonds are issued, then the price will be $β/γ$. Thus, unmatured bonds can have a higher price than a newly issued one-period bond, even though the two bonds have the same length of time to go to reach the maturity. The price differential exists because bonds of old vintages can command liquidity in the current goods market which newly issued bonds cannot.

The money growth rate ($γ$) affects real economic activities and welfare. Since the equilibrium quantity of goods, $q = f(1 + µ)$, is a decreasing function of $γ$, an increase in money growth reduces aggregate output and welfare. The nominal interest rate rises with $γ$. In contrast, the size of the open market operation, measured by $z$, has no effect on real activities or the nominal interest rate under given money growth rate. An increase in $z$ increases the bond-money ratio, which is

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7 Another way to express the positive discount on newly issued bonds is that one unit of current money is equivalent to one unit of money in the next period plus the liquidity value in the current goods market, while one unit of newly issued bonds is equivalent to one unit of money in the next period.
\[ b/m = zm_{-1}/m = z/\gamma. \] An increase in \( z \) also increases the price level.\(^8\)

3. Legal Restriction in the Goods Market

Now I introduce a legal restriction in the goods market to examine how the restriction affects bonds’ liquidity and other equilibrium properties.

3.1. Trades and Decisions

The legal restriction forbids the use of bonds as a means of payments for goods in a fraction \( g \in (0,1) \) of matches. In other matches the buyers can use both money and bonds as payments, as in the previous section. An example of how the restriction is enforced is that government agents participate in a fraction \( g \) of the matches and they accept only money as payments. A trade is called a restricted trade if the legal restriction is enforced and an unrestricted trade otherwise.

The legal restriction represents a matching shock to a household. I formulate this shock as follows: With probability \( g \) all members of the household will be located in restricted matches and with probability \( 1 - g \) they will be all located in unrestricted matches. This shock is realized after the household has already chosen the portfolio allocation but before the markets open in the period (see Figure 1). This particular way of modelling the matching shock implies that in each period, a household experiences either restricted or unrestricted trades, but not both. This simplification is not critical for the analytical results.\(^9\) Also, the shocks are independent across households and over time. In each period, there are a fraction \( g \) of households in restricted trades and a fraction \( 1 - g \) in unrestricted trades.

I allow the taste for the goods in restricted trades to be different from the taste for the goods in unrestricted trades. The utility of consumption is \( \theta^i u(c^i) \) for good \( i \), where \( i = n \) indicates an unrestricted trade and \( i = g \) a restricted trade. Normalize \( \theta^n = 1 \) and let \( \theta^g = \theta \ (> 0) \). When \( \theta = 1 \), tastes are symmetric over the two types of goods.

The cost function of production is \( \psi(.) \) for both types of trades. To ease exposition, I adopt the specification \( \psi(q) = q^\Psi \) in the remainder of this paper, where \( \psi_0 > 0 \) and \( \Psi > 1 \) are constants.

For the representative household’s decisions, let \( (q^n, x^n) \) be the quantities of goods and assets a buyer proposes in an unrestricted trade, and \( (q^g, x^g) \) the quantities in a restricted trade. The asset constraints in the two types of trades are:

\[ x^n \leq \frac{am + lb}{1 - \sigma}, \quad (3.1) \]

\(^8\)To confirm this, notice that the asset constraint binds in the goods market. Then, the price level of goods is

\[ p = \frac{am + b}{(1 - \sigma)q} = m \frac{1 + z(1 - \beta)/\gamma}{(1 - \sigma)q}. \]

\(^9\)See Appendix D for an alternative method of modelling the matching shock.
\[ x^b \leq \frac{am}{1 - \sigma}. \]  

The seller’s surplus is zero in each trade, and so
\[ x^i = \psi(q^i)/\Omega^m, \ i = n, g. \]

The representative household’s decision problem becomes:
\[
(PH') \ v(m, b) = \max \left\{ g \left[ \theta u(c^g) - \alpha \sigma (1 - \sigma) \psi(Q^g) + \beta v(m^g_{1+1}, b_{+1}) \right] + (1 - g) \left[ u(c^n) - \alpha \sigma (1 - \sigma) \psi(Q^n) + \beta v(m^n_{1+1}, b_{+1}) \right] \right\},
\]

where \( c^i = \alpha \sigma (1 - \sigma) q^i \) is consumption of goods obtained from type-\( i \) trades, with \( i \in \{g, n\} \).

The constraints in the goods market are (3.1) and (3.2). The constraints in the bonds market are still \( b^a \geq 0 \) and (2.4). The law of motion of unmatured bonds is still (2.12). Since the household chooses the amount of purchase of newly issued bonds \( d \) before the matching shock is realized, \( b_{+1} \) is independent of the realization of the shock. The household’s future money holding, conditional on the realization \((i)\) of the matching shock in the current period, is:
\[
m^i_{+1} = m - Sd + S^a \left[ (1 - l) b - b^u \right] + \alpha \sigma (1 - \sigma) \left( X^i - x^i \right) + (lb + b^u) + L_{+1}.
\]

Because \( X^i = x^i \) in all symmetric equilibria, \( m^g_{+1} = m^n_{+1} \), and so the household’s money holding is also independent of the realization of the matching shock. Thus, I will suppress the superscripts \((g, n)\) on \((m, \omega^m, \Omega^m)\).

Let \( \lambda^n \) be the shadow price of the asset constraint in an unrestricted trade, (3.1), and multiply \( \lambda^n \) by \( \alpha \sigma (1 - \sigma)(1 - g) \) before incorporating into the maximization problem. Similarly, let \( \lambda^g \) be the shadow price of the asset constraint in a restricted trade, (3.2), and multiply \( \lambda^g \) by \( \alpha \sigma (1 - \sigma) g \).

I adapt the optimal conditions in previous sections into the current economy, as follows:

(i) For \( q^i \):
\[
\theta^i u'(c^i) = (\omega^m + \lambda^i) \frac{\psi'(q^i)}{\Omega^m}, \ i = g, n. \tag{3.3}
\]

(ii) For \((a, l)\):
\[
\alpha \sigma [g \lambda^g + (1 - g) \lambda^n] = \rho \quad \text{if } a \in (0, 1), \tag{3.4}
\]
\[
\alpha \sigma (1 - g) \lambda^n + \omega^m = (\omega^m + \rho) S^a \quad \text{if } l \in (0, 1). \tag{3.5}
\]

In each condition, the variable attains the lowest value in the specified domain if the equality is replaced by “<”, and the highest value if “>”.

(iii) The optimal condition for \( d \) is still (2.10), and for \( b^u \) still (2.12).

(iv) For \((m, b)\) (envelope conditions):
\[
\omega^m_{-1}/\beta = \omega^m + (1 - a) \rho + a a \sigma [g \lambda^g + (1 - g) \lambda^n], \tag{3.6}
\]
\[
\omega^b_{-1}/\beta = (1 - l)(\omega^m + \rho) S^u + l [\omega^m + \alpha \sigma (1 - g) \lambda^n]. \tag{3.7}
\]

A symmetric equilibrium can be defined similarly to the one before. Again focus on equilibria with \( a > 0 \) and constant values \((\omega^m_{-1} m/\beta, \omega^b_{-1} b/\beta)\). Then, the conditions for \( a \) and \( d \), i.e., (3.4) and (2.10), hold with equality.
3.2. Equilibrium

The equilibrium can be classified according to whether $\lambda^n$ and $\lambda^g$ are positive or zero, i.e., whether the asset constraints bind in trades. Under $\gamma > \beta$, at least one of $\lambda^n$ and $\lambda^g$ must be positive.\footnote{If $\lambda^g = \lambda^n = 0$, then the envelope condition for money implies $\omega_1^m = \beta \omega^m$, which can hold only if $\gamma = \beta$.} Thus, there are three cases. Before listing these cases, notice the following facts. First, $\lambda^n > 0$ implies $l = 1$ because, if $l < 1$, then the optimal conditions for $l$ and $b^u$ (i.e., (3.5) and (2.12)) will imply $\lambda^n = 0$. Second, $\lambda^n = 0$ implies $S = (\beta/\gamma)^2$. To see this, note that the envelope condition for unmatured bonds (i.e., (3.7)) implies $\omega^b_1 = \beta \omega^m$ when $\lambda^n = 0$, and then the envelope condition for money and the optimal condition for newly issued bonds yield the specified bond price. Third, if $\lambda^g = 0$, then $\omega^b = \omega^m$ from the envelope conditions, and so $S = \beta/\gamma$. Finally, (3.3) solves for $\lambda^i$, which can be combined with (3.6) to solve for the quantities $(q^n, q^g)$ in the form $q = f(k)$, where $f(.)$ is defined in (2.15). The three cases of the equilibrium are listed below.

(i) Case PS (Perfect Substitutability): $\lambda^g = 0$ and $\lambda^n > 0$. In this case, $l = 1$, $S = \beta/\gamma$, $a = 1 - z \beta/\gamma$, $q^n = q^n_1$ and $q^g = q^g_2(\theta)$, where

$$q^n_1 \equiv f \left( 1 + \frac{\mu}{1-g} \right), \quad q^g_2(\theta) \equiv f \left( \frac{1}{\theta} \right).$$ (3.8)

(ii) Case IS (Imperfect Substitutability): $\lambda^g > 0$ and $\lambda^n > 0$. In this case, $l = 1$, $(\beta/\gamma)^2 < S < \beta/\gamma$, $1 - z \beta/\gamma < a < 1 - z (\beta/\gamma)^2$, $q^n = q^n_2(a)$ and $q^g = q^g_3(a, \theta)$, where\footnote{To obtain the equations below, substitute $\omega^b$ from (3.7) into $S = \omega^b/(\omega^m + \rho)$ and note from (3.6) that $\omega^m + \rho = \omega^m/\beta = \gamma \omega^m/\beta$. This generates $S = (\beta/\gamma)^2 [1 + \alpha \sigma (1 - g) \lambda^n / \omega^m]$. Next, note that $S = (1-a)/z$ from the market clearing condition for newly issued bonds. Combining these two equations for $S$ and using (3.3) to eliminate $\lambda^n$ yields $q^n = q^n_2(a)$. Then, using (3.3) to eliminate $(\lambda^n, \lambda^i)$ in (3.6) and substituting $q^n$ yields $q^g = q^g_3(a, \theta)$. Finally, (3.11) comes from dividing the binding constraints in restricted and unrestricted trades and using the facts that $l = 1$ in this case and $b/m = z/\gamma$.}

$$q^n_2(a) \equiv f \left( 1 + \frac{D(a)}{1-g} \right), \quad q^g_3(a, \theta) \equiv f \left( \frac{1}{\theta} \left( 1 + \frac{\mu - D(a)}{g} \right) \right),$$ (3.9)

$$D(a) = \frac{1}{\alpha \sigma} \left[ \frac{1-a}{z} \left( \frac{\gamma}{\beta} \right)^2 - 1 \right].$$ (3.10)

Also,

$$\frac{\psi(q^n_2)}{\psi(q^n_2)} = 1 + \frac{z}{\gamma a}.$$ (3.11)

(iii) Case BS (Bad Substitutability): $\lambda^g > 0$ and $\lambda^n = 0$. In this case, $0 \leq l < 1$, $S = (\beta/\gamma)^2$, $a = 1 - z (\beta/\gamma)^2$, $q^n = q^n_3$ and $q^g = q^g_3(\theta)$, where

$$q^n_3 \equiv f(1), \quad q^g_3(\theta) \equiv f \left( \frac{1}{\theta} \left( 1 + \frac{\mu}{g} \right) \right).$$ (3.12)
In Case PS, money and unmatured bonds are perfect substitutes in the goods market, i.e., \( \omega^b = \omega^m \). The legal restriction in the goods market does not bind, because the marginal utility of goods obtained from a restricted trade is so low that the buyer does not spend all the money in that trade. Money and unmatured bonds both yield liquidity, but only in unrestricted trades. There is a positive discount on newly issued two-period bonds, but the discount arises entirely from the one-period separation between the bonds market and the goods market.

Case BS is opposite to Case PS, in the sense that the asset constraint binds in a restricted trade but not in an unrestricted trade. Unmatured bonds do not yield liquidity in the goods market and so they are poor substitutes for money. A positive discount on unmatured bonds is necessary for the equilibrium, which induces a deeper discount on newly issued two-period bonds. Thus, the (two-period) bond price is lower than in Case PS. Moreover, since unmatured bonds do not yield liquidity, a household is indifferent at the margin about sending more unmatured bonds into either market. There are a range of values of \( l \) that are consistent with equilibrium. That is, unmatured bonds may or may not be perfectly liquid in this case.

Case IS is between Cases PS and BS. In Case IS, unmatured bonds are perfectly liquid in the goods market, as in Case PS, but they are not perfect substitutes for money. The substitutability is imperfect because, as a result of the legal restriction, unmatured bonds only yield liquidity in restricted trades while money yields liquidity in both restricted and unrestricted trades. Because of the imperfect substitutability, unmatured bonds are discounted, but this discount is smaller than in Case BS. Similarly, the positive discount on newly-issued two-period bonds is smaller than in Case BS but greater than in Case PS.\(^{12}\)

The term structure of interest rates also differs in these cases. In Case BS, the yield curve is flat, as the price of two-period bonds is equal to the square of the price of one-period bonds. In Cases PS and IS, however, unmatured bonds yield liquidity in the goods market, and so the yield curve is negatively sloped. The slope is steeper in Case PS than in Case IS, because unmatured bonds yield higher liquidity in Case PS.

To determine when each case occurs, define the following numbers and functions:

\[
A_1 = \psi^{-1} \left( 1 + \frac{z}{\gamma - z\beta} \right), \quad (3.13)
\]

\[
\Theta_1 = \frac{\psi' (q_1^n / A_1)}{w' (\alpha \sigma (1 - \sigma) q_1^n / A_1)}, \quad (3.14)
\]

\[
A_3 (l) = \psi^{-1} \left( 1 + \frac{lz}{\gamma - z\beta^2 / \gamma} \right), \quad (3.15)
\]

\[
\Theta_3 (l) = \left( 1 + \frac{\mu}{g} \right) \frac{\psi' (q_3^n / A_3 (l))}{w' (\alpha \sigma (1 - \sigma) q_3^n / A_3 (l))}. \quad (3.16)
\]

\(^{12}\)In both Cases PS and IS, no unmatured bond appears in the bonds market. As a result, the price of unmatured bonds in the bonds market is indeterminate and can exceed the price of newly issued one-period bonds. In Case BS, the price of unmatured bonds is \( S^n = \beta / \gamma \).
Note that $q_3^n > q_1^n$ and $q_3^g < q_1^g$ for all $\mu > 0$ (i.e., for $\gamma > \beta$). Also, $A_3(l) \geq 1$ and $\Theta'_3(l) < 0$ for all $l \in [0, 1]$ and $z \leq (\gamma/\beta)^2$. For all $z \leq \gamma/\beta$, $A_1 > A_3(l)$, $0 < \Theta_1 < 1$, and $\Theta_3(l) > \Theta_1$.

I prove the following proposition in Appendix B:

**Proposition 3.1.** An equilibrium exists for all $\theta \geq 0$ and $0 < z < (\gamma/\beta)^2$, and it is characterized by one of the three cases listed above. Case PS occurs for $z < \gamma/\beta$ and $\theta \leq \Theta_1 (< 1)$. Case BS occurs for $\theta \geq \Theta_3(l)$. Case IS occurs for $\Theta_1 < \theta < \Theta_3(1)$ if $z < \gamma/\beta$, and for $\theta < \Theta_3(1)$ if $z \geq \gamma/\beta$. The equilibrium is unique when $\theta \leq \Theta_3(1)$. When $\theta > \Theta_3(1)$, there are a continuum of equilibria (Case BS) that differ in the value of $l$ but have the same values of $(q^n, q^g, S, a)$.

![Diagram](image-url)

Case PS: Unmatured bonds are perfect substitutes for money;
Case IS: Imperfect substitutes; Case BS: Bad substitutes.

Figure 2

Figure 2 depicts the cases in the space of $(\theta, z)$. To find how the equilibrium changes with the taste parameter $\theta$, let me fix $z$ at a level lower than $\gamma/\beta$. When $\theta$ is very low, the economy is in Case PS. Unmatured bonds are perfectly liquid in the goods market and are perfect substitutes for money. An increase in $\theta$ in this region has no effect on the fraction of money allocated to the goods market, $a$, the fraction of liquid bonds, $l$ ($= 1$), the discount on newly issued bonds, and the quantity of goods traded in an unrestricted trade, $q^n$. The amount of money and the quantity of goods traded in a restricted trade increase with $\theta$.

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13If $z$ is fixed at a value in $[\gamma/\beta, (\gamma/\beta)^2)$, Case PS never occurs. For such a level of $z$, first Case IS, and then Case BS, occurs as $\theta$ increases.

15
When $\theta$ increases above the critical level $\Theta_1$, the equilibrium switches into Case IS. When $\theta$ increases in this case, unmatured bonds become less and less substitutable for money, and so newly issued bonds are discounted more and more heavily. Households increase the fraction ($a$) of money allocated to the goods market and increase the quantities of goods obtained in both restricted and unrestricted trades. The fraction of unmatured bonds that are liquid in the goods market remains at $l = 1$ in Case IS.

When $\theta$ increases further above the critical level $\Theta_3(1)$, the equilibrium switches into Case BS. Newly issued two-period bonds are discounted more heavily and the quantities ($a, q^n$) are higher than in other cases. The fraction of liquid bonds becomes indeterminate. The lower bound of this fraction decreases when $\theta$ increases, and so the set of equilibria enlarges. The price of new bonds ($S$) and the quantities ($a, q^n$) remain constant, while $q^g$ increases. Finally, when $\theta > \Theta_3(0)$, all values of $l$ in $[0, 1]$ are consistent with the equilibrium.

Unmatured bonds circulate in the goods market when tastes for restricted goods are not strong, which contrasts with the result in Shi (2002) that even an arbitrarily small legal restriction drives matured bonds out of the goods market. When tastes become more biased toward the goods in restricted trades, unmatured bonds become less liquid. When $\theta > \Theta_3(0)$, unmatured bonds can be completely illiquid in the equilibrium, because they do not generate liquidity service in the goods market. In this case, unmatured bonds are simply a store of value. Since $\Theta_3(0) = 1 + \mu/g$, unmatured bonds are more likely to become completely illiquid when the money growth rate is lower or when the fraction of trades that are restricted is higher.

### 3.3. Effects of Monetary Policy

I examine two kinds of monetary policy in this subsection, an open market operation and an increase in the money growth rate.

Consider first a tightening open market operation, modelled as an increase in $z$ which is accompanied by monetary transfers that maintain the money growth rate constant. When unmatured bonds are either perfect substitutes (Case PS) or bad substitutes (Case BS) for money in the goods market, the operation has no effect on the quantities of goods traded ($q^n, q^g$), and hence no effect on real consumption and output. The price of newly issued bonds does not change either. The larger supply of new bonds is met by an increase in the amount of money taken to the bonds market. That is, $a$ falls. Because newly issued bonds are sold at a discount and such bonds are redeemed for one unit of money, the total stock of nominal assets, $(m + b)$, rises.

When unmatured bonds are perfect substitutes for money (Case PS), the tightening operation increases the total “moneyness” in the goods market, $(am + b)$; Thus, the price level rises. When unmatured bonds are bad substitutes for money (Case BS), however, the price level is determined by the amount of money in the goods market, rather than by the sum of money and bonds. Because the tightening operation shifts money from the goods market to the bonds market, the
price level falls.\textsuperscript{14}

The effects of the open market operation are quite different in Case IS, where unmatured bonds are neither perfect nor poor substitutes for money. In contrast to Cases PS and BS, in Case IS the shift of money from the goods market to the bonds market does not offset the higher supply of bonds; In fact, the response of $a$ is ambiguous. The bond price falls and the nominal interest rate rises. Also in contrast to Cases PS and BS, the quantities of goods change. Since both money and unmatured bonds can relax the binding asset constraint in an unrestricted trade, the presence of a larger amount of these assets (as a result of the operation) increases the quantity of goods exchanged in an unrestricted trade. However, the quantity of goods exchanged in a restricted trade decreases. Thus, the tightening operation shifts consumption from the goods in restricted trades to the goods in unrestricted trades. The price level responds to the tightening operation ambiguously, because of the ambiguous response of $a$.

Now consider an increase in the money growth rate, $\gamma$, through lump-sum transfers. In all cases, a higher money growth rate reduces the real money balance and increases the inflation rate. In Cases PS and BS, a higher money growth rate reduces the bond price and increases the fraction of money in the goods market. The quantity of goods in a trade with a binding asset constraint falls, while the quantity of goods in a trade with a non-binding asset constraint does not change. In Case IS, the quantity of goods traded in an unrestricted trade falls, while other variables ($S, a, q^n$) all respond ambiguously to the higher money growth rate.

4. Effects of the Legal Restriction

It is natural to interpret the fraction of restricted trades ($g$) as the coverage of the legal restriction. However, this interpretation is correct only in the case $\theta = 1$. When $\theta \neq 1$, an increase in $g$ does more than just increasing the coverage of the legal restriction – it also replaces one set of goods by another set of goods which the household values differently.\textsuperscript{15} To interpret $g$ as the coverage of the legal restriction, I will restrict the analysis in this section to the case $\theta = 1$, i.e., the case where tastes are symmetric over goods.

It is convenient to express the existence of the equilibrium in terms of $g$. Notice that Case PS never occurs when $\theta = 1$, since it requires $\theta \leq \Theta_1 < 1$. Case BS occurs if $\Theta_3(1) \leq 1$ and Case IS occurs for $\Theta_3(1) > 1$. To write these conditions in terms of $g$, define $G_3$ by

\[
G_3 = \mu \left[ \frac{\psi''(q^n_3/A_3(1))}{\psi'(q^n_3/A_3(1))} - 1 \right].
\]  

\textsuperscript{14}To verify these effects, note the following facts. In Case PS, the binding asset constraint (in an unrestricted trade) yields $\omega^m = (1 - \sigma)\psi(q^n_3)/(a + z/\gamma)$, where $a = 1 - z\beta/\gamma$. The price of goods in an unrestricted trade is $p^n = (a + z/\gamma)m/[(1 - \sigma)q^n_3]$, and in restricted trades $p^g = \psi(q^n_3)/(q^n_3 \omega^m)$. In Case BS, the binding asset constraint (in a restricted trade) yields $\omega^m = (1 - \sigma)\psi(q^n_3)/a$. The price of goods in a restricted trade is $p^g = am/[(1 - \sigma)q^n_3]$, and in an unrestricted trade $p^g = \psi(q^n_3)/(q^n_3 \omega^m)$.

\textsuperscript{15}To illustrate this point, suppose that nominal bonds do not exist in the economy. Then, the legal restriction does not have any effect on the allocation. However, an increase in $g$ still changes the allocation when $\theta \neq 1$. 17
The following corollary can be verified:

**Corollary 4.1.** There exists \( \gamma_3 \in (\beta, \infty) \) such that \( G_3 < 1 \) iff \( \gamma < \gamma_3 \).\(^{16} \) When \( \theta = 1 \), the equilibrium is either Case IS (i.e., imperfect substitutability), which occurs for \( g < \min\{G_3, 1\} \), or Case BS (i.e., poor substitutability), which occurs iff \( G_3 \leq g < 1 \). In both cases, \( l > 0 \) for all \( g < 1 \); that is, there are always a positive fraction of unmatured bonds that circulate in the goods market. Moreover, \( q^n > q^0 \).

The legal restriction affects the quantities of goods traded. Examine Case BS first. In this case, an increase in \( g \) increases the quantity of goods traded in a restricted match and has no effect on the quantity in an unrestricted match. To explain these effects, recall that money generates liquidity service (i.e., the \( \lambda \)'es) by relaxing the asset constraint in matches. The expected value of such liquidity service is \( \alpha \sigma g \lambda^g \) in Case BS. When \( g \) increases, money relaxes the binding money constraint in more trades than before and so generates higher liquidity service. This amount of liquidity service, which serves as the non-pecuniary return to money, must be equal to \((\gamma/\beta - 1)\) in the steady state. To maintain the equality, \( \lambda^g \) must fall. That is, the gap between the marginal utility of consumption and the marginal cost of production in a restricted trade must fall. This requires the quantity of goods in a restricted trade to rise. The quantity of goods in an unrestricted trade does not change because it maintains \( \lambda^n = 0 \) in Case BS.

In Case IS, the asset constraint binds in an unrestricted trade as well as in a restricted trade, and the expected non-pecuniary return to money is \( \alpha \sigma [g \lambda^g + (1 - g) \lambda^n] \). As in Case BS, an increase in \( g \) increases the expected non-pecuniary return to money, because the asset constraint is more binding in a restricted trade than in an unrestricted trade; The quantity of goods traded in a restricted match must increase to restore such non-pecuniary return to the steady state level. In contrast to Case BS, however, the quantity of goods traded in an unrestricted match also responds to the increase in \( g \). This response is ambiguous. On the one hand, \( q^n \) tends to fall, so as to reduce the gap \((\lambda^g - \lambda^n)\) and to help restoring the non-pecuniary return to money to the steady state level. On the other hand, \( q^n \) tends to rise as a result of the increase in the fraction \((a)\) of money that the household allocates to the goods market.

In general, an increase in \( g \) tends to reduce the gap between the quantities of goods traded in the two types of matches, \((q^n - q^0)\). When this occurs, an increase in \( g \) smoothens consumption between the two states of the matching shock.

Now I turn to the effect of the legal restriction on social welfare. Let me measure welfare by the steady state utility level and express it as a function of \( g \):

\[
v^*(g) = \frac{1}{1 - \beta} \left\{ g \left[ u(c^g) - \alpha \sigma (1 - \sigma) \psi(q^g) \right] + (1 - g) \left[ u(c^n) - \alpha \sigma (1 - \sigma) \psi(q^n) \right] \right\}.
\]

\(^{16}G_3\) is an increasing function of \( \gamma \), with \( G_3 = 0 \) at \( \gamma = \beta \), and \( G_3 > 1 \) at \( \gamma \to \infty \). So, \( \gamma_3 \) is well-defined, and \( G_3 < 1 \) iff \( \gamma < \gamma_3 \).
An increase in $g$ affects social welfare in two ways. One is by shifting trades from unrestricted ones to restricted ones; This trade-shifting effect always reduces welfare, because a restricted trade generates a lower social surplus than an unrestricted trade does. The other welfare effect of $g$ is to reduce the gap between $q^n$ and $q^g$, as discussed above; This effect increases welfare because it smoothens consumption between the two states of the matching shock. The overall effect of $g$ on welfare depends on which of these two effects is stronger.

The relative strength of the two effects of $g$ on welfare changes when $g$ increases gradually from 0 to 1, which produces a U-shaped response of welfare to the coverage of the legal restriction. When $g$ is close to 0, the (negative) trade-shifting effect dominates and so welfare falls with an increase in $g$. When $g$ is large, the (positive) consumption-smoothing effect dominates and so welfare increases with an increase in $g$. At the two extreme values, $g = 0$ and 1, the welfare level is the same; i.e., $v^*(0) = v^*(1)$. This is because the quantity of goods in each trade is the same in the two extreme cases; that is, $q^g$ at $g = 1$ is equal to $q^n$ at $g = 0$.\footnote{To see this, note that in both Cases IS and BS, the following equation holds:

$$g \left(1 - g\right)^{\frac{\alpha}{1 - \delta} - 1} + g^{\frac{\alpha}{1 - \delta}} = 1 + \mu = \frac{u'(c_0)}{\Psi'(q_0)}$$

where $q_0$ is the quantity of goods in each trade when there is no legal restriction and $c_0 = \alpha \sigma (1 - \sigma) q_0$ is the corresponding level of consumption. Then, $g \to 0$ implies $c^n \to c_0$, and $g \to 1$ implies $c^g \to c_0$.}

Formally, the following proposition holds (see Appendix C for a proof).

**Proposition 4.2.** Assume $\theta = 1$. For any given $g \in (0, 1)$, there exists $\gamma_2 > \beta$ such that, if $\beta < \gamma \leq \gamma_2$, then $v^*(g) < v^*(0)$. However, there exists a sub-interval of $(0, 1)$ for $g$ in which $v^*(g)$ is increasing in $g$. That is, $v^*(g_2) > v^*(g_1)$ may hold for $1 > g_2 > g_1 > 0$.

The same results seem to hold for all feasible money growth rates, but the analytical proof is difficult. Instead, I consider the following example:

**Example 4.3.** $u(c) = \frac{c^{1-\delta} - 1}{1-\delta}$ and, as assumed in this section, $\psi(q) = q^\Psi$. Choose $\beta = 0.995$, $z = 0.05$, $\alpha = 1$, $\sigma = 0.5$, $\Psi = 2$, $\delta = 4$ and $\gamma = 1.05$. The large value of $\delta$ is chosen to favor the positive welfare effect of $g$ in smoothing consumption between the realizations of the matching shock. The money growth rate is large relative to the discount factor.

Figure 3a shows how the quantities of goods traded in matches respond to $g$ and Figure 3b shows the response of $DV$, which is the welfare gain from $g > 0$ in comparison with $g = 0$. The quantity of goods traded in a restricted match always increases with $g$, as discussed above. The quantity of goods traded in an unrestricted match increases with $g$ when $g$ is not too large and then remains constant when $g$ is large. This change in the pattern of $q^n$ occurs when the equilibrium switches from Case IS to Case BS. As $g$ increases from 0 to 1 gradually, welfare first falls and then rises. That is, increasing the coverage of the legal restriction reduces welfare
when the coverage is narrow, but increases welfare when the coverage is already wide. However, 
v^*(g) < v^*(0) = v^*(1) for all g ∈ (0, 1). Thus, an economy with a partial coverage of the legal 
restriction has lower welfare than an economy with either no coverage or complete coverage of 
the legal restriction.

Figure 3a Quantity of goods traded in a match

Figure 3b Welfare gain from the legal restriction

Here is a good place to compare my results with those in Rocheteau (2002). Using a similar 
search model, Rocheteau has obtained the same result $v^*(0) = v^*(1)$ as I did. In contrast to
my analysis, however, he finds that the legal restriction can improve welfare over an economy without the restriction. There are several differences between his model and mine that might cause the difference in the welfare result. For example, the bonds market is decentralized in his model, but centralized in my model; the goods market and the bonds market open sequentially in his model, but simultaneously in my model. However, the most important difference is his assumption that a household brings all bonds to the goods market; i.e., he sets \( l = 1 \) exogenously. Because the optimal choice of \( l \) can be less than 1 (see Case BS), his assumption implies that the price level can be too high in his model, which creates an artificially large gap \( (q^n - q^g) \). As a result, an increase in the coverage of the legal restriction can improve welfare by reducing this gap substantially. This result does not occur in my model because \( l \) is endogenous.

5. Liquidity of Bonds, Interest Rates and Welfare

In this section I conduct the following analysis. Suppose that the government imposes a physical restriction that makes a fraction \( (1 - L) \) of bonds illiquid after the households have purchased the bonds. Call \( L \) the degree of liquidity of the bonds. I compare economies that differ in \( L \) and, in particular, I analyze how welfare and interest rates change with bonds’ liquidity.

To simplify the formulas, I will adopt the special functional forms of \( u(c) \) and \( \psi(q) \) given in Example 4.3. However, the parameter \( \theta \) is not restricted to 1.

The equilibrium with an exogenous \( L \) has three cases, which resemble those in the economy where the liquidity of bonds is endogenous. To specify these cases, define

\[
\theta_1 = \left(1 + \frac{\mu}{1 - g}\right)^{-1} \left(1 + \frac{zL}{\gamma - (z^{\beta^2/\gamma})(1 + \alpha\sigma\mu L)}\right)^{\frac{1 - \delta}{\gamma L} - 1},
\]

\[
\theta_3 = \left(1 + \frac{\mu}{g}\right) \left(1 + \frac{zL}{\gamma - z^{\beta^2/\gamma}}\right)^{\frac{1 - \delta}{\gamma L} - 1}.
\]

By adapting Proposition 3.1, I can establish the following Lemma (the proof is omitted).

**Lemma 5.1.** In an economy with an exogenous \( L > 0 \), the equilibrium is one of the following three cases. If \( 0 < \theta \leq \theta_1 \) (Case PS), then \( q^n = q^n_1 \) and \( q^g = q^g_1 \); If \( \theta \geq \theta_3 \) (Case BS), then \( q^n = q^n_3 \) and \( q^g = q^g_3 \); If \( \theta_1 < \theta < \theta_3 \) (Case IS), then

\[
q^n = f \left(1 + \frac{D(a)/L}{1 - g}\right), \quad q^g = f \left(\frac{1}{\theta} \left(1 + \frac{\mu - D(a)/L}{g}\right)\right),
\]

\[
\frac{\psi(q^n)}{\psi(q^g)} = 1 + \frac{zL}{\gamma a}.
\]

Here, \( q^n_1, q^g_1, q^n_3, \) and \( q^g_3 \) are given by (3.8) and (3.12), and \( D(a) \) by (3.10). In all cases, the price of newly issued (two-period) bonds is

\[
S = \left(\frac{\beta}{\gamma}\right)^2 \left[1 + \alpha\sigma L(1 - g) \left(f^{-1}(q^n) - 1\right)\right]. \quad (5.1)
\]
The above characterization also applies to the economy with an exogenous $L = 0$, except that in Case IS the quantities of goods traded in matches are $q^n = q^g = f \left( \frac{1+\mu}{1-g+g\theta} \right)$.

The degree of bonds’ liquidity affects the region in which each of the three cases occurs. Because the critical levels $\theta_1$ and $\theta_3$ decrease with $L$, Case PS is less likely to occur and Case BS more likely to occur when $L$ is larger. That is, when a larger fraction of bonds circulate in the goods market, nominal bonds are less likely to be good substitutes for money. However, notice that the real allocation is independent of $L$ in Cases PS and BS. That is, the liquidity of nominal bonds does not matter if $\theta$ is either sufficiently small or sufficiently large.

Most of the differences between an economy with $L > 0$ and an economy with $L = 0$ occur in Case IS. First, when Case IS is the equilibrium in one economy, it may not be the equilibrium in the other economy. Second, the quantities of goods traded in matches depend on $L$ and hence are different in the two economies.

To see these differences clearly, consider first the special case $\theta = 1$. Then, Case IS is the equilibrium in the economy with $L = 0$, but the equilibrium with $L > 0$ can be either Case IS or Case BS. The quantity of goods is the same in an unrestricted trade as in a restricted trade (i.e., $q^n = q^g$) when $L = 0$, but $q^n > q^g$ when $L > 0$. Moreover, when $L$ increases, $q^n$ increases and $q^g$ decreases, and so the gap $(q^n - q^g)$ widens. When $\theta = 1$, this widening gap increases the difference between marginal utilities of consumption in the two types of trades; Thus, welfare is likely to be lower with liquid bonds than with (completely) illiquid bonds. By continuity, this welfare comparison also holds for $\theta$ around 1. The following proposition states the result formally: \footnote{The proof is omitted because it is very similar to the proof of Proposition 4.2.}

**Proposition 5.2.** For any given $g \in (0, 1)$, there exist a value $\gamma_A > \beta$ and a neighborhood $[\theta_A, \theta_B]$ which contains the value 1 in its interior such that, if $\beta < \gamma \leq \gamma_A$ and $\theta \in [\theta_A, \theta_B]$, then welfare is lower with liquid bonds than with (completely) illiquid bonds. Moreover, $q^n > q^g$ in this region of parameter values.

Next, I use numerical examples to examine the equilibrium in a wider range of values of $\theta$. Set $\beta = 0.995$, $\gamma = 1.005$, $g = 0.2$, $z = 0.05$, $\alpha = 1$, $\sigma = 0.5$, $\Psi = 2$ and $\delta = 2$. For $L > 0$, write the solutions for the variables as functions of $(\theta, L)$ and denote them with lower-case letters. For $L = 0$, write the solutions as functions of $\theta$ and denote them with capital-case letters. Let $dV(\theta, L) = v(\theta, L) - V(\theta)$ be the welfare gain of an economy with $L > 0$ relative to an economy with $L = 0$. The results are shown in Figures 4a through 4d.
$DV(i, j) = dV(0.85 + 0.01i, 0.01 + 0.05j)$; The axes on the plane are $i = 0 - 40 (\theta = 0.85 - 1.25)$ and $j = 0 - 20 (L = 0.01 - 0.99)$.

Figure 4a Welfare gain from liquid bonds

Figure 4b Welfare gain under two particular values of $L$
Figure 4c Quantity of goods in an unrestricted trade

Figure 4d Quantity of goods in a restricted trade

Figure 4a plots the welfare gain from liquid bonds for a range of values of $\theta$ and $L$. Confirming the above discussion, Figure 4a shows that, when $\theta$ is either small (e.g. $\theta < 0.85$) or large (e.g. $\theta > 1.25$), the welfare level is the same with liquid bonds as with illiquid bonds. When $\theta$ is close to 1, welfare is lower with liquid bonds than with illiquid bonds. However, there is an interval of values of $\theta$, bounded below 1, in which welfare is higher with liquid bonds than with illiquid bonds. Figure 4b illustrates these welfare effects of liquid bonds for two particular degrees of
bonds’ liquidity, $L = 0.25$ and $0.95$. It seems that, when bonds’ liquidity is higher, welfare is reduced by the liquidity in a wider interval of values of $\theta$, and the interval in which welfare is increased by the liquidity shifts to lower values of $\theta$.

Figures 4c and 4d illustrate the quantities of goods traded in matches under three degrees of bonds’ liquidity, $L = 0, 0.25$ and $0.95$. For $\theta$ around 1, an increase in bonds’ liquidity increases the quantity of goods traded in an unrestricted match, reduces the quantity of goods traded in a restricted match, and hence widens the gap $(q^n - q^g)$. As explained above, this widening gap reduces expected utility when $\theta$ is near 1, and the explanation applies also to the case $\theta > 1$. However, when $\theta$ is at some distance below 1, goods from unrestricted trades are valued more highly than goods from restricted trades. In this case, the widening gap $(q^n - q^g)$ reduces the gap in the marginal utility of consumption between the two states. This explains why welfare increases with bonds’ liquidity for such values of $\theta$.

To understand why $q^n$ and $q^g$ respond to $L$ in the way depicted in Figures 4c and 4d, note that the equilibrium is Case IS when $\theta$ is near 1. In Case IS, the asset constraint binds in both an unrestricted trade and a restricted trade. When more bonds can be used in the goods market to purchase goods, they relax the asset constraint in an unrestricted match, and hence increase the quantity of goods traded in such a match. At the same time, the influx of such liquid bonds reduces the real value of nominal assets. The fall in the real value of money makes the asset constraint in a restricted trade more binding, because money is the only asset that can be used to purchase goods in such a trade. Thus, the quantity of goods in a restricted trade falls.

So far I have compared two economies which differ in the exogenous degree of bonds’ liquidity. However, even when the households can choose the degree of bonds’ liquidity, the potential role of illiquid bonds in improving welfare will still exist when $\theta$ is near 1. This is because Case IS is the equilibrium when $\theta$ is near 1 and, in Case IS, the household’s optimal choice of $l$ is 1. That is, the equilibrium degree of bonds’ liquidity is 1, while the socially optimal degree is 0.

The welfare-improving role of illiquid bonds is similar to that in Kocherlakota (2001). In particular, the legal restriction implies a matching shock that resembles the taste shock in Kocherlakota. In both models, illiquid bonds can improve welfare by improving the smoothness in the marginal utility of consumption across different states of the shock. Aside from the differences in the details of the two models, however, an important contrast lies in the substitutability between nominal bonds and money. In Kocherlakota’s model, (unmatured) bonds are perfect substitutes for money when they are perfectly liquid. This is not necessarily so in the current model (see Proposition 5.2). In Case IS, in particular, bonds can be perfectly liquid, and yet they are imperfect substitutes for money due to the legal restriction. Since Case IS is the case in which illiquid bonds can improve welfare, my model partially disentangles the link between the welfare-improving role and perfect substitutability of liquid bonds.

Also in contrast with Kocherlakota, I conduct the welfare analysis for the entire spectrum of
bonds’ liquidity, rather than for the two extreme cases \( L = 0 \) and \( 1 \) only. For a fixed \( \theta \), it is possible that welfare changes non-monotonically with the degree of liquidity. When \( \theta = 0.95 \), for example, welfare in the above numerical example will first rise and then fall when \( L \) increases.

By examining all possible values of \( L \), I can also find how the nominal interest rate changes with bonds’ liquidity in general. Express the bond price \((S)\) and the corresponding nominal interest rate \((r)\) as functions of \((\theta, L)\). The following corollary holds.

**Corollary 5.3.** For any given \( \theta \), \( r(\theta, 0) \geq r(\theta, L) \) for all \( L \in (0, 1) \), and the inequality is strict if (completely) illiquid bonds improve welfare. However, \( r(\theta, L_1) > r(\theta, L_2) \) can hold for \( L_1 > L_2 > 0 \).

**Proof.** From (5.1) it is evident that \( S(\theta, 0) \leq S(\theta, L) \) for all \( L > 0 \), and so \( r(\theta, 0) \geq r(\theta, L) \). The inequality for \( r \) is strict when illiquid bonds improve welfare, because \( q^n < f(1) \) then. For \( L_1 > L_2 > 0 \), the numerical result below shows that \( r(\theta, L_1) > r(\theta, L_2) \) is possible. QED

When illiquid bonds improve welfare, the nominal interest rate is higher with illiquid bonds than with liquid bonds. In this sense, a higher nominal interest rate is associated with higher welfare, as Kocherlakota (2001) puts it. Illiquid bonds are discounted more heavily than liquid bonds in order to compensate for the loss of liquidity.

The negative relationship between the nominal interest rate and bonds’ liquidity is not general. Between two economies in which bonds have interior degrees of liquidity, a lower degree of liquidity can be associated with a lower nominal interest rate. This is illustrated in Figures 4e and 4f, with the parameter values used above. When \( \theta \) is large, the interest rate remains constant with respect to \( L \), as shown by the plateau in Figure 4e (the equilibrium is Case BS). When \( \theta \) is small, the interest rate decreases monotonically with \( L \), as shown by the sliding portion in Figure 4e. When \( \theta \) is fixed at an intermediate level, the interest rate first decreases and then increases as bonds’ liquidity increases. Figure 4f depicts these patterns for three particular levels of \( \theta \).

To see why the interest rate can respond non-monotonically to an increase in bonds’ liquidity, let me examine the bond price given by (5.1). An increase in \( L \) affects the bond price in two ways. One is to increase the probability with which a bond can be used to relax the asset constraint in an unrestricted trade; this effect increases the bond price. The other way is that \( L \) tends to increase the quantity of goods traded in an unrestricted trade, \( q^n \). This effect reduces the bond price, because it reduces the amount of liquidity service that unmatured bonds can generate by relaxing the asset constraint in an unrestricted trade. For low values of \( L \), the first effect dominates the second, in which case the bond price increases and the nominal interest rate decreases with \( L \). For high values of \( L \), however, the second effect can dominate the first effect, in which case the bond price decreases and the interest rate rises with \( L \).

Similarly, the welfare level is not always higher in the economy where bonds are less liquid. In the above numerical example with \( \theta = 1 \), it can be shown that the welfare level decreases...
monotonically with $L$ for all values of $L$. However, Figure 4f shows that the interest rate first falls and then rises with $L$ when $\theta = 1$. Thus, in this particular example, welfare and the nominal interest rate are positively related to each other when bonds are not very liquid, but negatively related when bonds are very liquid.

$$R(i, j) = r(0.85 + 0.01i, 0.01 + 0.05j);$$ The axes on the plane are $i = 0 – 40 (\theta = 0.85 – 1.25)$ and $j = 0 – 20 (L = 0.01 – 0.99)$

Figure 4e Dependence of two-period interest rate on $(\theta, L)$
6. Conclusion

In this paper I construct a hybrid model of a decentralized goods market and a centralized bonds market. The two markets are separated as in a typical model of limited participation (e.g., Lucas, 1990). In a fraction of matches in the goods market, a legal restriction prevents the use of bonds as the means of payments. I show that nominal bonds can become illiquid endogenously when tastes are strong for the goods in restricted trades; When such tastes are not very strong, bonds are liquid but may or may not be perfect substitutes for money.

This result can be construed as an answer to the question why unmatured bonds do not circulate in the goods market as widely as money. In particular, the legal restriction does not have to cover all trades in order to make unmatured bonds completely illiquid. However, the legal restriction cannot be negligible, either. Since an arbitrarily small legal restriction is sufficient to make matured bonds illiquid (see Shi, 2002), this paper shows that the illiquidity of unmatured bonds is fundamentally different from the illiquidity of matured bonds. To make unmatured bonds completely illiquid without any legal restriction, one must appeal to other differences between money and bonds that I have abstracted from the current model.

This paper provides some novel conclusions on the dependence of interest rates and welfare on the legal restriction and the liquidity of bonds. Welfare has U-shaped dependence on the coverage of the legal restriction. That is, as the coverage increases, welfare first decreases and then increases. Eliminating the legal restriction altogether increases welfare. Also, making bonds completely illiquid increases welfare and the nominal interest rate when tastes are nearly symmetric over the goods. However, between two economies in which bonds have interior degrees of liquidity, it is possible that welfare can be higher when bonds are more liquid, that welfare can be higher when the nominal interest rate is lower, and that the interest rate can be higher when bonds are more liquid.

The current model is useful for analyzing monetary policy. Because the bonds market here is the deterministic version of Lucas’s (1990) model, one can use the current model to re-examine the issues that have been analyzed in the framework of limited participation. The advantage of the current model is that it does not impose the cash-in-advance constraint in the goods market. Monetary policy has effects that are different from those in a cash-in-advance model. For example, a permanent increase in the size of the open market operation can have real effects (see subsection 3.3), which are absent in Lucas’s model.\textsuperscript{19} To see how the predictions are quantitatively different from a standard model of limited participation, one needs to work out a stochastic version of the current model first. This is left for future research.

\textsuperscript{19}In a model with spatial separation, Williamson (2002) also makes the point that the so-called liquidity effect in a standard model of limited participation is not robust to the introduction of private money.
References


A. Proof of Proposition 2.1

First, I show \( S^u \leq 1 \), suppose \( S^u > 1 \) to the contrary. Since \( \rho = \alpha \sigma \lambda \), the net marginal gain from allocating unmatured bonds to the goods market, as opposed to allocating them to the bonds market, is \( (\omega^m + \rho)(1 - S^u) \) (the difference between the two sides of (2.11)). \( S^u > 1 \) implies \( l = 0 \), and so the supply of unmatured bonds in the bonds market is positive. The clearing condition for such bonds requires (2.12) to hold, i.e., \( S^u = \omega^m / (\omega^m + \rho) \leq 1 \). A contradiction.

The above argument also implies that \( l = 1 \) if \( S^u < 1 \), and \( l \in [0, 1] \) if \( S^u = 1 \). Thus, \( l + (1 - l)S^u = 1 \) for all \( l \in [0, 1] \). Using this result and the fact \( \rho = \alpha \sigma \lambda \), I can rewrite the envelope conditions (2.14) and (2.13) to show that \( \omega^b / \beta = \omega^m + \alpha \sigma \lambda = \omega^m / \beta \). Thus, \( \omega^b = \omega^m \), as stated in the Lemma.

Second, I show \( S = \beta / \gamma \), \( a = 1 - z \beta / \gamma \), and \( S^u \geq \beta / \gamma \). Because \( d > 0 \) in equilibrium, (2.10) holds. Substituting \( \omega^b = \omega^m \) into (2.10) yields: \( 1 / S = 1 + \alpha \sigma \lambda / \omega^m \). From (2.13) I have: \( 1 + \alpha \sigma \lambda / \omega^m = \omega^m / \beta \omega^m \). Since the value of money, \( \omega^m / \beta \), is constant, then \( \omega^m / \omega^m = \omega^m / \beta \omega^m \). Combining these results, I get \( S = \beta / \gamma \). Since \( a = 1 - zS \) by (2.4), then \( a = 1 - z \beta / \gamma \). The fact \( b^u \geq 0 \) requires \( S^u \geq \omega^m / (\omega^m + \rho) = \beta / \gamma \), where the inequality comes from the optimal condition for \( b^u \) (see (2.12)) and the equality from (2.13).

Third, I show \( l = 1 \). Suppose \( l < 1 \), to the contrary. In this case, the equilibrium requires \( S^u = 1 \) (see the above proof). Since \( b^u = (1 - l)b > 0 \) in this case, (2.12) holds. With \( S^u = 1 \), this condition requires \( (\gamma / \beta) = 1 + \alpha \sigma \lambda / \omega^m = 1 / S^u = 1 \). This cannot hold under the maintained assumption \( \gamma > \beta \).

Fourth, I solve for \( q \). By the definition of the equilibrium, the value \( \omega^m / \beta \) is constant over time, and so \( \omega^m / \omega^m = \gamma \). Also, (2.8) implies \( \lambda / \omega^m = u'(c) / u'(q) - 1 \). Using this result to eliminate \( \lambda \) in (2.13) and noting \( c = \alpha \sigma(1 - \sigma)q \), I have \( q = f(1 + \mu) \).

Finally, the equilibrium exists under the restriction \( z \in (0, \gamma / \beta) \) (and \( \gamma > \beta \)). For all \( \gamma > \beta \), \( \mu > 1 \) and so \( q < f(1) \). This implies that the asset constraint binds in the goods market, as it is required. Also, because \( a = 1 - z \beta / \gamma \), the condition \( 0 < z < \gamma / \beta \) is necessary and sufficient for \( a \in (0, 1) \). QED

B. Proof of Proposition 3.1

Consider Case PS first. Since \( \lambda^g = 0 \) in this case, then \( S = \beta / \gamma \) as argued in the text, which leads to \( a = 1 - z \beta / \gamma \). Clearly, \( a > 0 \) iff \( z < \gamma / \beta \). Also, \( \lambda^u > 0 \) implies \( l = 1 \), as argued in the text. Since \( \lambda^g = 0 \), \( \lambda^u > 0 \) and \( l = 1 \), I have \( 1 + b / \beta m \leq \psi(q^u) / \psi(q^g) \) from the asset constraints in restricted and unrestricted trades. Substituting \( b / m = z / \gamma \) and \( a = 1 - z \beta / \gamma \), this condition becomes \( q^u(\theta) \leq q^u / A_1 \), where \( A_1 \) is defined by (3.13). Substituting \( q^u(\theta) \) from (3.8), this condition is equivalent to \( \theta \leq \Theta_1 \), where \( \Theta_1 \) is defined in (3.14).
Next, consider Case BS. Since $\lambda^n = 0$ in this case, then $S = (\beta/\gamma)^2$ as argued in the text, which leads to $a = 1 - z(\beta/\gamma)^2$. Then, $a > 0$ iff $z < (\gamma/\beta)^2$. Also, since $\lambda^g > 0$ and $\lambda^n = 0$, the following conditions hold:

$$\frac{am}{1-\sigma} = \frac{\psi(q^g)}{\omega^m}$$

$$\text{and} \quad \frac{am + lb}{1-\sigma} \geq \frac{\psi(q^n)}{\omega^m}.$$  

Dividing these two conditions yields $1 + \frac{lb}{am} \geq \frac{\psi(q^n)}{\psi(q^g)}$. Substituting $b/m = z/\gamma$ and $a = 1 - z(\beta/\gamma)^2$, the condition becomes $q_3^g(\theta) \geq q_3^g/A_3(l)$, where $A_3(l)$ was defined in (3.15). Substituting $q_3^g(\theta)$ from (3.12), this condition is equivalent to $\theta \geq \Theta_3(l)$, where $\Theta_3(l)$ is defined in (3.16).

Finally, consider Case IS. Since the case requires $\lambda^g, \lambda^n > 0$, and $\lambda^l/\omega^m = u'(c^i)/\psi'(q^i) - 1$ for $i \in \{n, g\}$, it is necessary that $u'(c^i) > \psi'(q^i)$ for $i = n, g$. With (3.9), these requirements are equivalent to $0 < D(a) < \mu$. It can be readily verified that $D(a) > 0$ is equivalent to $a < 1 - z(\beta/\gamma)^2$, and $D(a) < \mu$ to $a > 1 - z\beta/\gamma$. Since $a > 0$, the necessary conditions for Case IS are

$$\max\{0, 1 - z\beta/\gamma\} < a < 1 - z(\beta/\gamma)^2. \quad (B.1)$$

Clearly, $z < (\gamma/\beta)^2$ is required. Maintain these conditions. Then, for given $a$, $q_2^g(a)$ and $q_2^g(a, \theta)$ given by (3.9) are well-defined. Moreover, $q_2^g(a)$ is an increasing function and $q_2^g(a, \theta)$ a decreasing function of $a$.

Substituting $q_2^g(a)$ and $q_2^g(a, \theta)$ into (3.11), I obtain the following equation for $a$:

$$\frac{\psi(q_2^g(a))}{\psi(q_2^g(a, \theta))} - \left(1 + \frac{z}{\gamma a}\right) = 0.$$  

Temporarily denote the left-hand side as $LHS(a)$. Then, $LHS'(a) > 0$. If there is any solution for $a$ to this equation, the solution is unique. The necessary and sufficient conditions for existence of a solution are that $LHS(a) > 0$ when $a$ is at the upper bound in (B.1) and that $LHS(a) < 0$ when $a$ is at the lower bound in (B.1). When $a = 1 - z(\beta/\gamma)^2$, $q_2^g(a) = q_2^n$, $q_2^g(a, \theta) = q_2^g(\theta)$, and so $LHS(a) > 0$ at this upper bound of $a$ iff $q_2^g(\theta) < q_2^n/A_3(1)$. This condition is equivalent to $\theta < \Theta_3(1)$, where $\Theta_3(1)$ can be found by setting $l = 1$ in (3.16). Similarly, when $z < \gamma/\beta$, the lower bound of $a$ is $a = 1 - z\beta/\gamma$. At this lower bound, $q_2^g(a) = q_2^n$, $q_2^g(a, \theta) = q_2^g(\theta)$, and so $LHS(a) < 0$ iff $\theta > \Theta_1$, where $\Theta_1$ is defined by (3.14). When $\gamma/\beta < z < (\gamma/\beta)^2$, the lower bound of $a$ is 0. Since $0 < D(0) \leq \mu$, the numbers $q_2^n(0)$ and $q_2^g(0, \theta)$ are positive and finite, and so $LHS(0) = -\infty < 0$. Therefore, if $z < \gamma/\beta$, Case IS exists for $\Theta_1 < \theta < \Theta_3(1)$; if $\gamma/\beta < z < (\gamma/\beta)^2$, Case IS exists for $\theta < \Theta_3(1)$. QED

**C. Proof of Proposition 4.2**

The second part of the proposition is evident from the fact that $v^*(0) = v^*(1)$: If $v^*(g) \neq v^*(0)$ for some $g \in (0, 1)$, then there exists a sub-interval of values of $g$ in which $v^*(g)$ is increasing.
To show the first part of the proposition, write the quantity of goods traded in an unrestricted trade as $q^n = f(k^n)$ and in a restricted trade as $q^g = f(k^g)$, where $k^n = 1 + \frac{D(a)}{1-g}$ and $k^g = 1 + \frac{\mu}{g}$ in Case IS, and $k^n = 1$ and $k^g = 1 + \mu/g$ in Case BS. Notice that $k^n, k^g \geq 1$ in these cases and that $f'(k) < 0$. Define

$$W(k) = u(\alpha \sigma(1 - \sigma)f(k)) - \alpha \sigma(1 - \sigma)\psi(f(k)).$$

Then $(1 - \beta)v^*(g) = gW(k^g) + (1 - g)W(k^n)$. Also, $v^*(0) = v^*(1) = W(1 + \mu)/(1 - \beta)$.

Notice that $gk^g + (1 - g)k^n = 1 + \mu$. If $W(k)$ is concave, then

$$(1 - \beta)v^*(g) < W(1 + \mu) = (1 - \beta)v^*(0).$$

Thus, to show that $v^*(g) < v^*(0)$ for $g \in (0, 1)$, it suffices to show that there exists an interval $[1, k_0]$ such that $k^n, k^g \in [1, k_0]$ and $W(k)$ is concave in this region.

Compute $W''$:

$$W''(k) = \alpha \sigma(1 - \sigma) \left[\psi' f'' + (k - 1) \left(\psi'' f'^2 + \psi' f''\right)\right].$$

Because $f' < 0$, we have $W''(1) < 0$. Thus, there exists $k_0 > 1$ such that $W''(k) < 0$ for $k \in [1, k_0]$. Finally, we restrict $\gamma$ so that $k^n, k^g \in [1, k_0]$. For this purpose it suffices to restrict $k^g \leq k_0$, because $k^g > k^n$ in both Cases IS and BS. (In Case IS, in particular, (3.11) requires $q^n > q^g$, and so $f(k^n) > f(k^g)$, which in turn requires $k^g > k^n$.) A sufficient condition for $k^g \leq k_0$ in the two cases is $\mu \leq g(k_0 - 1)$. This condition can be written as $\gamma \leq \gamma_2 \equiv \beta [1 + \alpha \sigma g(k_0 - 1)].$

Clearly, $\gamma_2 > \beta$. QED
D. A Variation of the Model with the Legal Restriction

In this appendix, I assume that a household’s members experience the matching shocks independently, rather than that the members have the same realization of the shock as I assumed in section 3. This new assumption implies that a household experience both restricted trades and unrestricted trades in each period, where the fraction of restricted trades is $g$ and the fraction of unrestricted trades is $1-g$. Assume that the two types of goods can be aggregated to yield consumption as follows:

$$c = \alpha \sigma (1 - \sigma) [g \psi(q^g) + (1 - g)\psi(q^n)],$$

where $\theta > 0$ is the relative quality of goods obtained from restricted goods to goods obtained from unrestricted goods. The utility of consumption in each period is $u(c)$. To ease the task in this appendix, I will analyze only the case $\theta = 1$. (This rules out the case of perfect substitutability between unmatured bonds and money, like Case PS in section 3.)

As in section 3, a superscript $n$ indicates unrestricted trades and a superscript $g$ restricted trades. The representative household’s optimization problem is as follows:

$$v(m, b) = \max \{u(c) - \alpha \sigma (1 - \sigma) [g \psi(Q^g) + (1 - g)\psi(Q^n)] + \beta v(m+1, b+1)\}.$$}

Most constraints, including the asset constraints in the goods market, are the same as in section 3. The only exception is the law of money, which is

$$m+1 = m - Sd + S^u [(1 - l)b - b^u] + [lb + b^u] + L_{+1} + \alpha \sigma (1 - \sigma)(1 - g)(X^n - x^n) + \alpha \sigma (1 - \sigma)g(X^g - x^g).$$

The optimal conditions for $(q^i, a, l)$ and the envelope conditions for $(m, b)$ are the same as in section 3, with $\theta^i = 1$ for $i = g, n$. The optimal condition for $d$ is (2.10) and for $b^u$ (2.12).

Again focus on equilibria with $a > 0$ and constant values $(\omega_{-1}^m m/\beta, \omega_{-1}^b b/\beta)$. Then, the conditions for $a$ and $d$ (i.e., (3.4)) and (2.10)) hold with equality.

**Lemma D.1.** In all equilibria, $l > 0$, $\omega^b < \omega^m$, $q^g < q^n$, and $\lambda^g > \lambda^n (\geq 0)$. Moreover, if $l < 1$, then $\lambda^n = 0$.

**Proof.** I show the second part of the lemma first. If $l < 1$, then $(1 - l)b > 0$. Market clearing for unmatured bonds in the bonds market requires the optimal condition for $b^u$, (2.12), to hold with equality. This yields $S^u = \omega^m / (\omega^m + \rho)$. Then, the net gain from increasing $l$ is $\alpha \sigma (1 - g)\lambda^n$ (see (3.5)). This net gain must be zero in order for the choice $l < 1$ to be optimal. Thus, $l < 1$ implies $\lambda^n = 0$.

Next, I show that $q^g \leq q^n$. Suppose $q^g > q^n$, contrary to the lemma. Because (3.3) implies $\lambda^i / \omega^m = u'(c) / \psi'(q^i) - 1$ for $i = n, g$, then $\lambda^n > \lambda^g (\geq 0)$. Together with the supposition $q^g > q^n$, this implies

$$\frac{am + lb}{1 - \sigma} = \frac{\psi(q^n)}{\omega^m} < \frac{\psi(q^g)}{\omega^m} \leq \frac{am}{1 - \sigma}.$$
A contradiction.

In fact, (3.3) implies that \( q^g \leq q^n \) if and only if \( \lambda^g \geq \lambda^n \), and \( q^g < q^n \) if and only if \( \lambda^g > \lambda^n \).

Now I show \( l > 0 \). Suppose \( l = 0 \), contrary to the lemma. Because \( l < 1 \) in this case, the previous argument implies \( \lambda^n = 0 \). The envelope condition for money, (3.6), becomes \( \omega_{-1}^m / (\omega m) = 1 + \alpha \sigma g \lambda^g / \omega m \). This can hold (under \( \gamma > \beta \)) only if \( \lambda^g > 0 \), because \( \omega_{-1}^m / \omega m = \gamma \) in all stationary equilibria (where \( \omega_{-1}^m / \beta \) is constant over time). Because \( \lambda^g > 0 = \lambda^n \) in this case, \( q^g < q^n \). Then,

\[
\frac{am + lb}{1 - \sigma} \geq \frac{\psi(q^n)}{\omega m} > \frac{\psi(q^n)}{\omega m} = \frac{am}{1 - \sigma}.
\]

This contradicts the supposition \( l = 0 \).

With \( l > 0 \), I can show \( q^g < q^n \), which implies \( \lambda^g > \lambda^n (\geq 0) \) as shown before. It suffices to rule out \( q^g = q^n \). Suppose, to the contrary that \( q^g = q^n \), and so \( \lambda^g = \lambda^n \). Because \( l > 0 \), \( \lambda^g \) and \( \lambda^n \) must be zero; otherwise the asset constraints in a restricted trade and an unrestricted trade would lead to a contradiction. Under \( \gamma > \beta \), the result \( \lambda^g = \lambda^n = 0 \) violates the envelope condition for money. Therefore, \( q^g < q^n \).

Finally, I show \( \omega^b < \omega^m \). Because \( l > 0 \), (3.5) implies \( \alpha \sigma (1 - g) \lambda^n + \omega^m \geq (\omega^m + \rho) S^u \), where the inequality is strict only when \( l = 1 \). Thus, \( (1 - l)(\omega^m + \rho) S^u = (1 - l) [\alpha \sigma (1 - g) \lambda^n + \omega^m] \).

Substituting this into the envelope condition for unmatured bonds, (3.7), I get:

\[
\omega_{-1}^b / \beta = \omega^m + \alpha \sigma (1 - g) \lambda^n.
\]

The envelope condition for money, (3.6), can be rewritten as:

\[
\omega_{-1}^m / \beta = \omega^m + \alpha \sigma [g \lambda^g + (1 - g) \lambda^n].
\]

Thus, \( \omega_{-1}^b - \omega_{-1}^m = -\beta \alpha \sigma g \lambda^g < 0 \). QED

The features described in the above lemma are very similar to those of Cases IS and BS in Corollary 4.1. First, unmatured bonds circulate in the goods market, despite the legal restriction. Second, the fraction of liquid bonds may be equal to or less than one. Third, unmatured bonds are not perfect substitutes for money, because money can generate liquidity in restricted trades but unmatured bonds cannot. Fourth, because unmatured bonds are imperfect substitutes for money, they are discounted, and this discount induces a deeper discount on newly issued two-period bonds. Finally, bonds are discounted differently depending on the vintage and the length of maturity. In fact, the following inequalities hold as in Cases IS and BS:

\[
1 > S^u \geq \frac{\beta}{\gamma} > S
\]

where the inequality “\( \geq \)” becomes equality when \( l < 1 \).

There are two possible equilibria. One has \( 0 < l < 1 \), which resembles Case BS in section 3; the other has \( l = 1 \), which resembles Case IS. I examine them in turn.

In the equilibrium with \( 0 < l < 1 \), \( \lambda^n = 0 \). Substitute \((\lambda^n, \lambda^g)\) from (3.3). Then, the condition \( \lambda^n = 0 \) and the envelope condition for money yield:

\[
\psi'(q^n) = u'(c), \quad (D.1)
\]
\[
\mu = g \left[ \frac{u'(c)}{\psi'(q^n)} - 1 \right],
\]

(D.2)

where \( c = \alpha \sigma (1 - \sigma)(gq^n + (1 - g)q^m) \) and \( \mu \) is defined in (2.16). The above two conditions solve for \((q^n, q^g)\) jointly. The following proposition details the existence conditions:

**Proposition D.2.** There exist two constants \( \gamma_1 \) and \( z_1 \), where \( \gamma_1 \geq \beta \) and \( z_1 < (\gamma / \beta)^2 \), such that an equilibrium with \( 0 < l < 1 \) exists if and only if \( \gamma > \gamma_1 \) and \( z_1 < z < (\gamma / \beta)^2 \). Under these conditions, there is a continuum of equilibria that differ among themselves in the value of \( l \). In all such equilibria, real quantities are independent of \( l \). So are bonds prices and the fraction \( a \), which are given as follows:

\[
a = 1 - z(\beta / \gamma)^2, \quad S^u = \beta / \gamma, \quad \text{and} \quad S = (\beta / \gamma)^2.
\]

**Proof.** Because \( l < 1 \) in this equilibrium, \( S^u = \beta / \gamma \). Also, \( \lambda^n = 0 \) by Lemma D.1. The envelope condition for unmatured bonds, (3.7), yields \( \omega^b_{-1} = \beta \omega^m \). Since the price of newly issued bonds satisfies (2.10), then

\[
S = \frac{\omega^b}{\omega^m + \rho} = \frac{\beta \omega^m_{-1}}{\omega^m_{-1} / \beta} = \left( \frac{\beta}{\gamma} \right)^2.
\]

The second equality follows from \( \omega^b = \beta \omega^m_{-1} \) and the envelope condition for money, while the last equality from \( \omega^m_{-1} = \gamma \omega^m \) for all \( t \).

The market clearing condition for newly issued bonds implies \( a = 1 - z(\beta / \gamma)^2 \). Then, \( 0 < a < 1 \) if and only if \( z < (\gamma / \beta)^2 \). Maintain this condition. Then, (D.1) and (D.2) imply \( \psi'(q^n) = \left( 1 + \frac{\mu}{g} \right) \psi'(q^g) \). Thus,

\[
q^g = \psi'^{-1} \left( \frac{\psi'(q^n)}{1 + \mu / g} \right) \equiv G(q^n, \mu / g).
\]

Substituting this function, I can rewrite (D.1) as follows:

\[
\psi'(q^n) = u' \left( \alpha \sigma (1 - \sigma) \left[ gG(q^n, \mu / g) + (1 - g)q^n \right] \right).
\]

Temporarily denote the right-hand side of this equation as \( RHS(q^n) \). Clearly, \( RHS'(q^n) < 0 \). Since \( \psi'(q) \) is an increasing function, there is a solution for \( q^n > 0 \) if and only if \( \psi'(0) < RHS(0) \) and \( \psi'(\infty) > RHS(\infty) \). These conditions also guarantee the uniqueness of the solution. The second condition is clearly satisfied. The first condition is equivalent to:

\[
\mu > \mu_1 \equiv g \left[ \frac{\psi'(0)}{\psi'(\frac{u^{-1}(\psi'(0))}{\alpha \sigma (1 - \sigma) g})} - 1 \right].
\]

This is equivalent to \( \gamma > \gamma_1 \), where

\[
\gamma_1 \equiv \beta \cdot \max \{ 1, 1 + \alpha \sigma \mu_1 \}.
\]

Once \( q^n \) is solved, \( q^g \) is given by \( G(q^n, \mu / g) \). Clearly, \( q^n > q^g \) for all \( \gamma > \beta \). Note that the solutions for \((q^n, q^g)\) are independent of \( z \).

The solutions must satisfy \( 0 < l < 1 \). Because \( \lambda^g > 0 \) and \( \lambda^n = 0 \), I have:

\[
\frac{am + lb}{1 - \sigma} \geq \frac{\psi'(q^n)}{\omega^m}, \quad \frac{am}{1 - \sigma} = \frac{\psi'(q^g)}{\omega^m}.
\]
Using these conditions and the facts that $a = 1 - z(\beta/\gamma)^2$ and $m/b = \gamma/z$, I obtain:

$$ l \geq \left( \frac{\gamma}{z} - \frac{\beta^2}{\gamma} \right) \left[ \frac{\psi'(q^n)}{\psi'(q^\theta)} - 1 \right]. $$

Clearly, this implies $l > 0$ for all $\gamma > \beta$ and $z < (\gamma/\beta)^2$. For $l < 1$, it is sufficient that

$$ z > z_1 \equiv \left( \frac{\beta}{\gamma} \right)^2 + \frac{1}{\gamma \left[ \frac{\psi'(q^n)}{\psi'(q^\theta)} - 1 \right]}^{-1}. $$

(D.3)

Since $(q^n, q^\theta)$ are independent of $z$, this condition is well-specified.

There is no condition to pin down $l$, provided that it is bounded from below (see above).

QED

The requirement $\gamma > \gamma_1$ is weak. For example, if $\psi'(0) = 0$, then $\gamma_1 = \beta$. The requirement $z > z_1$ is to ensure that the bond-money ratio is sufficiently high so that the asset constraint does not bind in an unrestricted trade for some $l < 1$. The requirement $z < (\gamma/\beta)^2$ is to ensure $a > 0$. Since $S = S^2_w$ in these equilibria, the term structure of interest rates is flat.

The following results can be shown:

$$ \frac{dq^\theta}{d\gamma} < 0, \quad \frac{dq^n}{d\gamma} > 0, $$

$$ \frac{dq^\theta}{dg} > 0, \quad \frac{dq^n}{dg} \sim q^n - q^\theta - \frac{\mu \psi'(q^n)}{(1 + \mu)^2 \psi'(q^\theta)}. $$

Now turn to the equilibrium with $l = 1$. If $\lambda^n = 0$ in this case, then the equilibrium is the limit of previous equilibria with $l \to 1$. So, let us focus on $\lambda^n > 0$. Substituting $\lambda^n$ and $\lambda^\theta$ from (3.3), I have:

$$ \mu = g \left[ \frac{u'(c)}{\psi'(q^\theta)} - 1 \right] + (1 - g) \left[ \frac{u'(c)}{\psi'(q^n)} - 1 \right]. $$

(D.4)

The price of two-period bonds is

$$ S = \left( \frac{\beta}{\gamma} \right)^2 \left[ 1 + \alpha \sigma (1 - g) \left( \frac{u'(c)}{\psi'(q^n)} - 1 \right) \right]. $$

(D.5)

Since $\lambda^\theta \geq \lambda^n > 0$, $\frac{am + b}{1 - \sigma} = \frac{\psi(q^n)}{\psi(q^\theta)}$ and $\frac{am}{1 - \sigma} = \frac{\psi(q^\theta)}{\omega^n}$. Dividing these two equations, using $b/m = z/\gamma$, and substituting $a = 1 - zS$ from the market clearing condition for newly issued bonds, I get:

$$ 1 = \left\{ \frac{\gamma}{z} - \frac{\beta^2}{\gamma} \left[ 1 + \alpha \sigma (1 - g) \left( \frac{u'(c)}{\psi'(q^n)} - 1 \right) \right] \right\} \left[ \frac{\psi(q^n)}{\psi(q^\theta)} - 1 \right]. $$

(D.6)

With $c = \alpha \sigma (1 - \sigma)[gq^\theta + (1 - g)q^n]$, the two equations (D.4) and (D.6) solve $(q^n, q^\theta)$ jointly. Once this is done, (D.5) gives the price of two-period bonds and $a = 1 - zS$ determines $a$.

---

To derive this formula, note first that $S = \omega^n/\omega^m$. Next, the envelope condition for unmatured bonds implies $\omega^\theta = \beta \omega^n_2 \left[ 1 + \alpha \sigma (1 - g) \lambda_1^n/\omega^m_1 \right]$. Then, the formula of $S$ comes from substituting $\omega^m + \rho = \omega^m_1/\beta$ from the envelope condition for money, $\omega^m_1 = \gamma^2 \omega^m_1$ from the stationarity of the equilibrium, and $\lambda^n$ from (3.3).
Proposition D.3. If $0 < z < z_1$ and $\gamma > \beta$, there exists an equilibrium with $l = 1$. In this equilibrium, $S > (\beta/\gamma)^2$ and $\beta/\gamma \leq S^n < 1$.

Proof. The right-hand side of (D.4) is a decreasing function of $q^n$ and $q^g$. For $\gamma > \beta$, the equation solves for a relationship $q^g = Q_1(q^n)$, where $Q_1'(q) < 0$. The equilibrium described in the proposition requires $q^n > q^g = Q_1(q^n)$. Denote the solution to $q = Q_1(q)$ by $q_a$. Then, $q_a = f(1 + \mu)$, where $f$ is defined in (2.15). Clearly, $q^n > Q_1(q^n)$ iff $q^n > q_a$.

The equilibrium also requires $\psi'(q^n) < u'(c)$. Let $q_b$ be the value of $q^n$ that maintains $\psi'(q^n) = u'(c)$, where $c = \alpha\sigma(1 - \sigma)[gQ_1(q^n) + (1 - g)q^n]$. Then, by (D.4), $q_b$ is explicitly given by:

$$\frac{\psi'(q_b)}{\psi'(Q_1(q_b))} = 1 + \frac{\mu}{g}.$$ 

Also, $\psi'(q^n) < u'(c)$ iff $q^n < q_b$. Clearly, $q_b > q_a$. (Notice that $q_b$ is the value of $q^n$ in the equilibrium with $0 < l < 1$.)

Restrict $q^n \in (q_a, q_b)$. Denote the right-hand side of (D.6) as $RHS(q^n, q^g)$. Then, $q^n$ solves

$$1 = RHS(q^n, Q_1(q^n)).$$

Clearly, $RHS(q_a, Q_1(q_a)) = 0 < 1$. A sufficient condition for there to be a solution for $q^n \in (q_a, q_b)$ is $RHS(q_b, Q_1(q_b)) > 1$. Because $u'(c) = \psi'(q_b)$ by definition and $q_b$ is the solution for $q^n$ in the equilibrium with $0 < l < 1$, the sufficient condition is $z < z_1$, where $z_1$ is defined by (D.3). QED

When the supply of bonds is limited, households allocate all unmatured bonds to the goods market to trade for goods, despite the legal restriction. Because unmatured bonds generate liquidity in the goods market, their price can exceed the price of newly issued one-period bonds, as in the economy without the legal restriction. Also, the price of newly issued two-period bonds exceeds that in the economy without the legal restriction. The term structure of interest rates has a negative slope. These properties are similar to those of Case BS in Corollary 4.1.