Meetings with Costly Participation: Reply

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Abstract: This note corrects an error in an example in "Meetings with Costly Participation" (AER 90(4), 927-943). It characterizes the set of equilibria for the example under the assumptions in the paper, shows that in all the equilibria an interval of moderate positions is devoid of participants, and provides assumptions under which the result as originally stated is correct.

Our paper Osborne, Rosenthal, and Turner (2000) studies the strategic game in which each of a set of players chooses whether to participate in a decision-making process to select a policy, modeled as a point in $\mathbb{R}^\ell$. Each player i has a favorite policy x_i. The outcome of an action profile a is a compromise $m(a)$ among the favorite policies of the players j for whom a_j is "participate". Player i's payoff is $v(x_i - m(a)) - c$ if she participates and $v(x_i - m(a))$ if she does not, where for each $d \in \mathbb{R}^\ell$, $v(\alpha d)$ is decreasing in α for $\alpha \geq 0$. If $v(z)$ depends only on $\|z\|$, we say that v is symmetric, and (with a slight abuse of notation) denote its value $v(\|z\|)$.

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One of our main results (Proposition 2) is that if the function v is concave and symmetric then there are functions $\underline{z}$ and $\overline{z}$ (which we define explicitly[1]) such that in any Nash equilibrium in which $k \geq 2$ players participate, the distance between the compromise and the favorite position of any participant is at least $\underline{z}(k)$ and the distance between the compromise and the favorite position of any nonparticipant is at most $\overline{z}(k)$. We summarize this conclusion as saying that "Only players whose favorite positions are sufficiently far from the compromise participate" and "only players whose favorite positions are sufficiently close to the compromise do not participate" (p. 929). We study in detail an example in which the policy space is one-dimensional ($\ell = 1$) and the compromise $m(a)$ is the median of the participants' favorite positions.

Francesco De Sinopoli and Giovanna Iannantuoni (2004) make two points. First, they show that our characterization of the equilibria in the example (given in our Proposition 3) errs in claiming that under our stated conditions no equilibrium exists in which the number of participants is odd. Proposition 3* below provides a correct characterization of the equilibria under our stated assumptions; the subsequent corollary strengthens the assumptions to

[1]The definition of $\overline{\beta}(k)$ in the paper is erroneous. The condition $x \neq m(X)$ in the set over which the sup is taken should be replaced by the condition $x \neq m(X \setminus \{x\})$. (We are grateful to Francesco De Sinopoli and Giovanna Iannantuoni for pointing out that Proposition 2 fails under the definition of $\overline{\beta}(k)$ in the paper. Note that in the definitions of both $\overline{\beta}(k)$ and $\underline{\beta}(k)$, the set X ranges over all possible k-element subsets of the policy space $\mathbb{R}^\ell$. In particular, the positions in X are not restricted to be favorite positions of the players.) This change in the definition of $\overline{\beta}(k)$ requires that the proof for the case that j attends in Proposition 2 be modified as follows. Consider an equilibrium in which j attends. Then $x_j \neq m(Y \setminus \{x_j\})$ (given $c > 0$ and $v(z) \leq 0$ for all z). Now, if $\overline{\beta}(k) = \infty$ we have $\underline{z}(k) = 0$, so certainly $\|x_j - m(Y)\| \geq \underline{z}(k)$. If $\overline{\beta}(k)$ is finite, then $x_j \neq m(Y)$, so that by the definition of $\overline{\beta}(k)$ we have $\|x_j - m(Y \setminus \{x_j\})\| \leq \overline{\beta}(k)\|x_j - m(Y)\|$. (Note the following typographic errors in the proof of Proposition 2: y_j should be x_j in the first displayed equation and $\underline{\beta}(k)$ and $\overline{\beta}(k)$ should be interchanged throughout the proof.)

make the original characterization correct.

Second, De Sinopoli and Iannantuoni's example highlights the fact that when $\underline{z}(k) = 0$ or $\overline{z}(k) = \infty$, our interpretation of Proposition 2 as saying that all participants are "extremists" and all nonattendees are "moderates" is strained. Indeed, in any equilibrium of the example in which the number of participants is odd, the position of one participant is exactly equal to the compromise.

Proposition 3* below shows that all equilibria of games satisfying the assumptions of Proposition 3 nevertheless involve an interval of moderate positions devoid of participants. The interval is not centrally located among the set of participants, as it is for equilibria with an even number of participants, but lies on one side of the compromise. (Refer to Figure 1. In De Sinopoli and Iannantuoni's example, the interval lies between -11 and 1.)

In the paper we define $\overline{y}$ by the condition $-v(\frac{1}{2}\overline{y}) = c$. Define $\underline{y}$ to be the policy such that $v(\frac{1}{2}\underline{y}) - v(\underline{y}) = c$. (On page 934, the paper defines $\underline{y} = 2\underline{z}(k)$, without noting that when the compromise function is the median, $\underline{z}(k)$ is the same for all even values of k.) If v is strictly concave, then $\overline{y} > \underline{y}$. For any z with $0 < z \leq \frac{1}{2}\overline{y}$, we define $\Delta(z)$ to be the unique solution of the equation $v(\Delta(z)) - v(z + \Delta(z)) = c$. Extend the definition of Δ to all $z > 0$ by letting $\Delta(z) = 0$ if $z > \frac{1}{2}\overline{y}$, and define the function δ by $\delta(z) = z + \Delta(z)$ for $z > 0$. We have $\delta(z) = \frac{1}{2}\overline{y}$ for $z \geq \frac{1}{2}\overline{y}$, and δ is decreasing on $(0, \frac{1}{2}\overline{y})$.

PROPOSITION 3*: *Suppose that the policy space is one-dimensional, the compromise function is the median, the valuation function is strictly concave and symmetric, the list of all the players' favorite positions is symmetric, and the* *default policy is* 0*. An action profile with at least one participant is an equilibrium if and only if either*

- *the number of participants is even, the distance between the favorite positions* x_h *and* $x_i > x_h$ *of the two central participants* h *and* i *is at least* $\underline{y}$*, the distance between* x_{h+1} *and* x_{i-1} *is at most* $\overline{y}$ *if* $i \geq h+2$*, and every player whose position is less than* $x_h - \Delta(\frac{1}{2}(x_i - x_h))$ *or more than* $x_i + \Delta(\frac{1}{2}(x_i - x_h))$ *participates*

- *or the number of participants is odd and at least three and the positions* $x_h < x_i < x_j$ *of the three central participants satisfy* $|x_i - \frac{1}{2}(x_j + x_h)| \geq \frac{1}{2}\overline{y}$ *and either*

  · $x_i - x_h > x_j - x_i$ *and* $x_i - x_h \geq \Delta(\frac{1}{2}(x_j - x_i))$

  · *no player has a favorite position between* x_h *and* $x_i - \underline{y}$

  · *every player whose favorite position is at most* x_h *or greater than* $x_i + \delta(\frac{1}{2}(x_j - x_i))$ *participates*

  or

  · $x_j - x_i > x_i - x_h$ *and* $x_j - x_i \geq \Delta(\frac{1}{2}(x_i - x_h))$

  · *no player has a favorite position between* $x_i + \underline{y}$ *and* x_j

  · *every player whose favorite position is less than* $x_i - \delta(\frac{1}{2}(x_i - x_h))$ *or at least* x_j *participates.*

Equilibria in which the number of participants is odd, an example of which is illustrated in Figure 1, share qualitative properties with equilibria in which the number of participants is even: all players with sufficiently extreme positions participate and an interval of moderate positions is devoid of participants. Specifically, for an equilibrium with an odd number of participants in which x_h is further from x_i than is x_j, all players with favorite positions less than x_h or greater than $x_i+\delta(\frac{1}{2}(x_j-x_i))$ participate, and all players with favorite positions from x_h to x_i do not participate. Note that the participation/nonparticipation boundary is sharper on one side of the central participant for equilibria with an odd number of participants than it is for equilibria with an even number of participants. Note also that the lower bound on the length of the interval of moderate positions devoid of participants is larger for an equilibrium with an odd number of participants (it is[2] $\overline{y}$) than it is for an equilibrium with an even number of participants (for which it is $\underline{y}$).

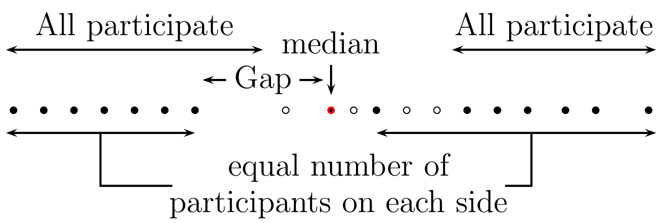

Figure 1. The structure of an equilibrium when the compromise function is the median, the valuation function is concave and symmetric, and the number of participants is odd. Each disk represents a participant's favorite policy and each circle represents a nonparticipant's favorite policy.

The following logic lies behind the conditions for an equilibrium with an odd number of pariticipants. If player i withdraws, the outcome changes from x_i to the mean of x_h and x_j, so that x_i has to be far enough from this mean to make player i's participation worthwhile. This condition implies that $x_h <$

[2]The conditions $x_h < x_i < x_j$ and $|x_i - \frac{1}{2}(x_j + x_h)| \geq \frac{1}{2}\overline{y}$ imply that the distance between x_i and the more distant of x_h and x_j exceeds $\overline{y}$.

$x_i - \underline{y}$. Suppose that x_h is further from x_i than is x_j. (If x_j is further from x_i than is x_j, symmetric arguments apply.) If player h withdraws, the outcome changes from x_i to the mean of x_i and x_j, so that for h's participation to be worthwhile, x_i has to be remote enough from x_h, given the distance between x_i and x_j. If a player, say k, were to have a favorite position between x_h and $x_i - \underline{y}$, then she would not participate (otherwise x_h would not be the position of the closest participant to i's left), in which case her switching to participation would change the outcome from x_i to $\frac{1}{2}(x_k + x_i)$, increasing her payoff. Now consider a player whose favorite position is less than x_h. If she does not participate, then her switching to participation changes the outcome from x_i to the mean of x_h and x_i; given that i's participation is optimal, this switch to participation increases her payoff. Finally, a nonparticipant whose favorite position exceeds $x_i + \delta(\frac{1}{2}(x_j - x_i))$ increases her payoff by switching to participation.

The fact that in any equilibrium in which the number of participants is odd either no player has a favorite position between x_h and $x_i - \underline{y}$ or no player has a favorite position between $x_i + \underline{y}$ and x_j (depending on which of x_h and x_j is further from x_i) leads to the observation that for some specifications of the individuals' favorite positions, no equilibrium exists in which the number of participants is odd.

COROLLARY: *If either (a) the players' favorite positions are equally-spaced or (b) the distance between every pair of adjacent positions is at most $\overline{y} - \underline{y}$, then the game has no Nash equilibrium in which the number of participants is odd.*

Proof. The proof of (a) is contained in the proof of Proposition 3 in the paper. Part (b) follows from the fact that in any equilibrium with an odd number of participants, the distance between the position x_i of the middle participant and the position of the more distant of her two neighbors, say x_h, exceeds $\overline{y}$, and any player whose position is between x_h and $x_i - \underline{y}$ must participate. $\square$

Proof of Proposition 3^*. The argument for an equilibrium with an even number of participants is in the paper. Consider an equilibrium with an odd number of participants.

We first argue that a strategy profile that satisfies the conditions in the result for the case in which $x_i - x_h \geq x_j - x_i$ is an equilibrium. The argument for the case in which $x_i - x_h \leq x_j - x_i$ is symmetric.

The outcome of such a strategy profile is x_i. If i withdraws then the outcome changes to $\frac{1}{2}(x_h + x_j)$, so that i's participation is optimal if

$$-c \geq v(|x_i - \tfrac{1}{2}(x_h + x_j)|)$$

or

$$|x_i - \tfrac{1}{2}(x_h + x_j)| \geq \tfrac{1}{2}\overline{y},$$

one of the conditions in the result. (As noted in footnote 2, this condition implies that $x_i - x_h > \overline{y}$ and hence $x_i - x_h > \underline{y}$.)

If j withdraws then the outcome changes from x_i to $\frac{1}{2}(x_h + x_i)$, so that j's participation is optimal if

$$v(|x_j - x_i|) - c \geq v(|x_j - \tfrac{1}{2}(x_h + x_i)|),$$

or

$$-c \geq v(|x_j - \tfrac{1}{2}(x_h + x_i)|) - v(|x_j - x_i|).$$

Now, $|x_j - \frac{1}{2}(x_h + x_i)| - |x_j - x_i| \geq |x_i - \frac{1}{2}(x_h + x_j)|$ because $|x_j - x_i| \leq |x_h - x_i|$. Thus the concavity of v and the fact that i cannot profitably withdraw means that j cannot profitably withdraw.

If h withdraws then the outcome changes from x_i to $\frac{1}{2}(x_i + x_j)$, so that her participation is optimal if

$$v(|x_i - x_h|) - c \geq v(|\tfrac{1}{2}(x_i + x_j) - x_h|),$$

or

$$v(|x_i - x_h|) - v(|\tfrac{1}{2}(x_i + x_j) - x_h|) \geq c,$$

or

$$|x_i - x_h| \geq \Delta(\tfrac{1}{2}|x_j - x_i|),$$

given the concavity of v. This condition also is given in the result.

Now consider a player whose favorite position is at most x_h. For a strategy profile that satisfies the conditions in the result, she participates. If she withdraws, the change in the outcome is the same as the change in the outcome when h withdraws. Given the concavity of v, her participation is thus optimal.

A player whose favorite position lies between $x_i - \underline{y}$ (which exceeds x_h) and x_i, like any player whose favorite position lies between x_h and x_i, does not participate in a strategy profile that satisfies the conditions in the result. If such a player, say k, switches to participation, the outcome changes from x_i to $\frac{1}{2}(x_k + x_i)$; given the definition of $\underline{y}$, the deviation does not make k better off.

Finally, any participant whose favorite position is at least x_j is not better off withdrawing for the same reason that j is not better off withdrawing, and a nonparticipant whose favorite position is at least x_j and at most $x_i + \delta(\frac{1}{2}(x_j - x_i))$ is not better off participating, given the definition of δ and the fact that her switching to participation changes the outcome from x_i to $\frac{1}{2}(x_i + x_j)$.

We now argue that any equilibrium in which the number of participants is odd satisfies the conditions in the result.

First we argue that no action profile in which a single player participates is an equilibrium. If the single participant in such a profile is i, then we need $v(|x_i|) \leq -c$ for i's participation to be optimal. In particular, $x_i \neq 0$. But then the player whose position is symmetric with x_i about 0 is better off switching to participation, given the strict concavity of v.

Now consider an action profile with at least three participants. Let i be the central participant, and h and j the participants closest to i on each side of x_i, with $x_h < x_i < x_j$. If $x_i - x_h = x_j - x_i$ then i's withdrawal does not affect the outcome, so that her participation is not optimal. Thus $x_i - x_h \neq x_j - x_i$. Suppose that $x_i - x_h > x_j - x_i$. (The case in which $x_j - x_i > x_i - x_h$ is symmetric.)

If i withdraws then the outcome changes to $\frac{1}{2}(x_h + x_j)$. Thus for i's participation to be optimal we require

$$v(|x_i - \tfrac{1}{2}(x_h + x_j)|) \leq -c$$

or

$$|x_i - \tfrac{1}{2}(x_h + x_j)| \geq \tfrac{1}{2}\overline{y}.$$

If h withdraws then the outcome changes from x_i to $\frac{1}{2}(x_i + x_j)$, so that for her participation to be optimal we require

$$v(|x_i - x_h|) - c \geq v(|\tfrac{1}{2}(x_i + x_j) - x_h|),$$

or

$$v(|x_i - x_h|) - v(|\tfrac{1}{2}(x_i + x_j) - x_h|) \geq c,$$

or

$$|x_i - x_h| \geq \Delta(\tfrac{1}{2}|x_j - x_i|).$$

Now consider a player, say k, whose favorite position is at most x_h. If this player does not participate, her switching to participation changes her payoff from $v(|x_i - x_k|)$ to $v(|\frac{1}{2}(x_h + x_i) - x_k|) - c$. Now, i's switch to nonparticipation changes i's payoff from $-c$ to $v(|x_i - \frac{1}{2}(x_h + x_j)|)$ and does not make her better off, so $-c \geq v(|x_i - \frac{1}{2}(x_h + x_j)|)$. Further, $|x_i - x_k| - |\frac{1}{2}(x_h + x_i) - x_k| > |x_i - \frac{1}{2}(x_h + x_j)|$, so that the strict concavity of v implies that $v(|x_i - x_k|) - v(|\frac{1}{2}(x_h + x_i) - x_k|) > v(0) - v(|x_i - \frac{1}{2}(x_h + x_j)|) = -v(|x_i - \frac{1}{2}(x_h + x_j)|)$. We conclude that $v(|x_i - x_k|) - v(|\frac{1}{2}(x_h + x_i) - x_k|) > c$, so that k's switching to participation increases her payoff. Thus every player whose favorite position is at most x_h participates in an equilibrium.

Now consider a player, say k, whose favorite position is between x_h and $x_i - \underline{y}$ (which exceeds x_h by an earlier argument). By assumption this player does not participate, because x_h is the favorite position of the participant closest to x_i on the left. If she switches to participation, then the outcome changes from x_i to $\frac{1}{2}(x_k + x_i)$, a distance greater than $\frac{1}{2}\underline{y}$. Thus by the definition of $\underline{y}$, player k is better off switching to participation. We conclude that for the configuration to be an equilibrium, no such player must exist.

Finally, consider a player, say k, whose favorite position exceeds $x_i + \delta(\frac{1}{2}(x_j - x_i))$. If she does not participate then her payoff is $v(|x_k - x_i|)$. If she switches to participation, her payoff becomes $v(|x_k - \frac{1}{2}(x_j + x_i)|) - c$. Thus the increase in her payoff if she switches to participation is

$$v(|x_k - \tfrac{1}{2}(x_j + x_i)|) - c - v(|x_k - x_i|),$$

which is positive because $x_k > x_i + \delta(\frac{1}{2}(x_j - x_i))$. Hence every such player participates in any equilibrium. $\square$

References

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