### Test 3 Solutions

These solutions are purposely detailed for your convenience

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**YOU CANNOT LEAVE THE ROOM IN THE LAST 10 MINUTES OF THE TEST**

**REMAIN SEATED UNTIL ALL TESTS ARE COLLECTED**

**IF YOU DETACH PAGES IT’S YOUR RESPONSIBILITY TO RE-STAPLE PAGES. GRADERS ARE NOT RESPONSIBLE FOR LOOSE PAGES**

**TIME: 1 HOUR AND 50 MINUTES**

<table>
<thead>
<tr>
<th>LAST NAME (AS IT APPEARS ON ROSI)</th>
<th>FIRST NAME (AS IT APPEARS ON ROSI):</th>
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</thead>
<tbody>
<tr>
<td>MIDDLE NAME (AS IT APPEARS ON ROSI)</td>
<td>U TORONTO ID # (AS IT APPEARS ON ROSI)</td>
</tr>
</tbody>
</table>

**PLEASE CIRCLE THE SECTION IN WHICH YOU ARE OFFICIALLY REGISTERED (NOT NECESSARILY THE SECTION YOU ATTEND)**

- MON 10 – 12
- MON 2 – 4
- TUE 10 – 12
- TUE 4 – 6
- WED 6 – 8

**SIGNATURE:** ________________________________

**SCORES**

<table>
<thead>
<tr>
<th>Question</th>
<th>Total Points</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>60</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>40</td>
<td></td>
</tr>
</tbody>
</table>

**Total Points = 100**

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**ONLY AID ALLOWED: A CALCULATOR**

**FOR YOUR THERE ARE TWO WORKSHEETS AT THE END OF THE TEST**

**GOOD LUCK!**

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1 Thanks: Asad Priyo

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S. Ajaz Hussain, Dept. of Economics, University of Toronto
Question 1 [60 Points. All parts worth 10 points each]

This question is based on the HBS case: The Prestige Telephone Company. For your convenience here is exhibit 1 from the case (figures below are for the Prestige Data services company January – March 2003)

<table>
<thead>
<tr>
<th></th>
<th>January 2003</th>
<th>February 2003</th>
<th>March 2003</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercompany Hours</td>
<td>206</td>
<td>181</td>
<td>223</td>
</tr>
<tr>
<td>Commercial Hours</td>
<td>123</td>
<td>135</td>
<td>138</td>
</tr>
<tr>
<td>Total Revenue Hours</td>
<td>329</td>
<td>316</td>
<td>361</td>
</tr>
<tr>
<td>Service Hours</td>
<td>32</td>
<td>32</td>
<td>40</td>
</tr>
<tr>
<td>Available Hours</td>
<td>223</td>
<td>164</td>
<td>143</td>
</tr>
<tr>
<td>Total Hours</td>
<td>584</td>
<td>512</td>
<td>544</td>
</tr>
</tbody>
</table>

In the case, the “commercial” price is $P_c = 800$/hour and the “intercompany” price is $P_i = 400$/hour:

\[
P_i = 400/hr. \\

C(Q) = TFC + TVC(Q) = 223,436 + 28Q

The Prestige Data Services’ cost function was estimated to be (here $Q$ is “hours of data services”):

The Prestige Data Services’ commercial inverse demand function in March 2003 was estimated to be:

\[
P(q) = 1,466 - 4.83q
\]
(a) Under the terms of a regulation ruling PDS’s intercompany billing are capped at an average of $82,000/month. What does this imply for the average intercompany hours that can be billed per month? Is Prestige Data Services abiding by or violating the terms of the ruling? Give a brief explanation.

Answer:
If intercompany revenues are capped at $82,000/month and prices are $400/hr then:

\[ P_i q_i = R_i \]

\[ 400 q_i = 82,000 \]

\[ q_i = 205 \]

This implies that intercompany hours should be 205 hours/month on average. Is PDS billing 205 hours on average each month? Look at:

<table>
<thead>
<tr>
<th>Exhibit 1: Prestige Telephone Company</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
</tr>
<tr>
<td>Intercompany Hours</td>
</tr>
<tr>
<td>January 2003</td>
</tr>
<tr>
<td>Commercial Hours</td>
</tr>
<tr>
<td>123</td>
</tr>
<tr>
<td>Total Revenue Hours</td>
</tr>
<tr>
<td>329</td>
</tr>
<tr>
<td>Service Hours</td>
</tr>
<tr>
<td>32</td>
</tr>
<tr>
<td>Available Hours</td>
</tr>
<tr>
<td>223</td>
</tr>
<tr>
<td>Total Hours</td>
</tr>
<tr>
<td>584</td>
</tr>
</tbody>
</table>

Notice that average intercompany hours from Jan – Mar 2003 were:

\[ \frac{206 + 181 + 223}{3} = \frac{610}{3} = 203.33 \]

Thus, PDS is abiding by the terms of the agreement.

(b) What kind of returns does PDS have for its “variable” inputs? Give a brief explanation based on the figures above (not below).

Answer:
The Prestige Data Services’ cost function was estimated to be (here \( Q \) is “hours of data services”):

\[ C(Q) = TFC + TVC(Q) = 223,436 + 28Q \]

Since this is a linear cost function PDS must have constant returns. In particular, notice that:

\[ AVC(q) = \frac{TVC(q)}{q} = \frac{28q}{q} = 28 \]
Since $AVC(q)$ is constant, PDS has constant returns.

(c) Recall that PDS has never earned profits. Calculate the breakeven number of commercial hours and the equation of the demand curve in the month when PDS breaks even. Show all calculations.

**Answer**

PDS sells data services to intercompany and commercial customers. Now profits are:

$$\Pi = \Pi_c + \Pi_i$$

$$\Pi = R_c + R_t - TVC_c - TVC_i - TFC$$

$$\Pi = P_c Q_c + P_i Q_i - AVC_c Q_c - AVC_i Q_i - TFC$$

$$\Pi = (P_c - AVC_c) Q_c + (P_i - AVC_i)Q_i - TFC$$

Breakeven commercial output is when:

$$\Pi = (P_c - AVC_c) (\text{Breakeven } Q_c) + (P_i - AVC_i)Q_i - TFC = 0$$

$$(P_c - AVC_c) (\text{Breakeven } Q_c) = TFC - (P_i - AVC_i)Q_i$$

Breakeven $Q_c = \frac{TFC - (P_i - AVC_i)Q_i}{P_c - AVC_c}$

Assuming $AVC_i = AVC_c = AVC = 28$ and $Q_i = 205$ we have:

$$\text{Breakeven } Q_c = \frac{223,436 - (400 - 28)205}{800 - 28} \approx 191$$

We can now compute the demand curve in the month when PDS breaks even. Assuming the “other factors” besides price are pushing the demand curve right we see that in the breakeven month, the demand curve has the same slope as the March 2003 demand curve:
Thus, in the breakeven month:

\[ P_c = a - 4.83 q_c \]

Since \( P_c = $800 \) and \( q_c = 192 \) we have:

\[ P_c = a - 4.83 q_c \]
\[ 800 = a - 4.83 (192) \]
\[ 800 + 4.83 (192) = a \]
\[ a = 1,727.36 \]

The commercial demand curve in the breakeven month is:

\[ P_c = 1,727.36 - 4.83q_c \]

(d) In general, is it easier for the “commercial division” to breakeven if PDS comprises of “commercial and intercompany divisions” versus if PDS comprises of just a “commercial division”? Assume the two scenarios have the same total fixed cost.

**Answer**

With two divisions we had:
Breakeven \( Q_c = \frac{\{TFC - (P_t - AVC_i)Q_i \}}{P_c - AVC_c} \)

Let’s compare this expression with the case of a single “commercial” division:

Breakeven \( Q_c = \frac{\{TFC \}}{P_c - AVC_c} \)

Notice that so long as \( P_t > AVC_i \) (i.e. the “other”, intercompany, division has positive contribution margin), then the commercial breakeven quantity with another “profitable” division is smaller than the breakeven quantity if the commercial division was by itself.

(e) [This part is independent of all other parts] Suppose that in March 2003, the government imposes a 10% excise tax on commercial data services. Assuming all commercial customers can be modeled by a single representative consumer with income \( $Y \) and utility function \( U(Q_1, Q_2) \) (where good 1 is commercial data services and good 2 is “everything else”) what is the marginal utility due to the excise tax on commercial data services? Please show all calculations and specifically state all assumptions.

Answer:

Before the excise has been imposed, the “representative” consumer, with an unknown income \( $Y \), consumes 138 hours of commercial data services and some unknown quantity of other goods. Let “everything else” be the base good so that \( P_2 = 1 \) and assume the consumer has a quasi-linear utility function of the type:

\[ U(Q_1, Q_2) = f(Q_1) + Q_2 \]

Here \( f(Q_1) \) is any function of good 1 such that \( f(0) = 0 \). In this case, the total utility of a bundle minus the income is the consumer surplus of good 1:
That is at bundle A:

\[ U(Q_1, Q_2) - Y = U(138, Q_2) - Y = CS = \frac{1}{2} (1,466 - 800)138 = \$45,954 \]
A 10% excise tax raises the price of commercial services to $880 and reduces demand to:

\[ P_c = 1,466 - 4.83q_c \]
\[ 880 = 1,466 - 4.83q_c \]
\[ 4.83q_c = 1,466 - 880 \]
\[ q_c = \frac{1,466 - 880}{4.83} \approx 121.33 \]

Thus, at bundle B:

\[ U(Q'_1, Q'_2) - Y = U(121.33, Q'_2) - Y = CS = \frac{1}{2} (1,466 - 880)121.33 \approx 35,550 \]
Therefore, the change in utility due to a 10% excise tax on commercial data services is:

\[ MU \text{ of a } 10\% \text{ excise tax on } Q_1 = \{ U(Q'_1, Q'_2) - Y \} - \{ U(Q_1, Q_2) - Y \} = U(Q'_1, Q'_2) - U(Q_1, Q_2) \]

\[ MU \text{ of a } 10\% \text{ excise tax on } Q_1 = U(Q'_1, Q'_2) - U(Q_1, Q_2) = $35,550 - $45,954 = -$10,404 \]

Notice this calculation does not require knowledge of the actual level of income, the exact utility function, or the amount of good 2 being consumed.
(f) Recall that the PDS’s “variable” inputs were quasi-variable power (denote by \( P \)) and quasi-variable labor (denote by \( L \)) while its “fixed” inputs were quasi-fixed power, quasi-fixed labor and all other inputs. Denote all “fixed” inputs as capital \( k \). Suppose PDS’s production function is:

\[
q = \alpha \frac{1}{P^\gamma} L^\gamma k^\gamma
\]

Assume \( a = k = 1 \) and \( \gamma > 0 \). Suppose the price of power is \( P_P \$/hour\), the price of quasi-variable labor is \( P_L = $30.25 \$/hour\) and the price of “capital” is \( P_K \). Given that \( AVC_P = $4 \$/hour\) and \( AVC_L = $24 \$/hour\) what is the price of quasi-variable power \( P_P \)? Hint: Solve the CMP

\[
\max_{P,L} -C = -P_P P - P_L L - P_K k \quad \text{s.t. } q = \alpha P^\alpha L^\beta k^\gamma
\]

and use the fact that \( AVC_P = $4 \$/hour\) and \( AVC_L = $24 \$/hour\). Show all calculations below.

**Answer:**
We are told that:

\[
AVC_P = 4
\]

This means that:

\[
\frac{TVC_P}{q} = \frac{P_P P}{q} = 4
\]

To find \( P_P \) we need to express the optimal demand for power \( P \) in terms of the parameters and (hopefully) solve for \( P_P \). This means we have to solve the CMP.

The total cost of \( q \) hours of data services is:

\[
C(q) = P_P P + P_L L + P_K k
\]

We could substitute values for some parameters now or we could work as long as possible in parametric form and then substitute numbers. We’ll do the latter so that you can see the algebra.

The Cost Minimization Problem (CMP) is:

\[
\min_{P,L} C(q) = P_P P + P_L L + P_K k \quad \text{s.t. } \alpha \frac{1}{P^\gamma} L^\gamma k^\gamma = q, P, L \geq 0
\]

Now: \( a = k = 1 \) so that:

\[
\min_{P,L} C(q) = P_P P + P_L L + P_K k \quad \text{s.t. } \alpha P^\gamma L^\gamma = q, P, L \geq 0
\]
Since the production function is of the Cobb-Douglas form we know that for \( q > 0 \) we must use some power and labor so that \( P, L > 0 \). As such, we can drop the non-negativity constraints:

\[
\min_{P, L} C(q) = P_p P + P_L L + P_K k \text{ s.t. } P^\frac{1}{7} L^\frac{6}{7} = q
\]

Now:

\[
\max_{P, L} -C(q) = -P_p P - P_L L - P_K k \text{ s.t. } P^\frac{1}{7} L^\frac{6}{7} = q
\]

Setup the Lagrangian:

\[
\max_{P, L} -C(q) = -P_p P - P_L L - P_K k \text{ s.t. } P^\frac{1}{7} L^\frac{6}{7} = q
\]

\[
\max_{P, L} -C(q) = -P_p P - P_L L - P_K k \text{ s.t. } P^\frac{1}{7} L^\frac{6}{7} = q
\]

\[
\max_{P, L} \lambda = -P_p P - P_L L - P_K k - \lambda [P^\frac{1}{7} L^\frac{6}{7} - q]
\]

The FOCs are:

\[
\frac{\partial \lambda}{\partial P} = -P_p - \frac{\lambda}{7} P^\frac{6}{7} L^\frac{6}{7} = 0
\]

\[
\frac{\partial \lambda}{\partial L} = -P_L - \frac{6 \lambda}{7} P^\frac{1}{7} L^{-\frac{1}{7}} = 0
\]

\[
\frac{\partial \lambda}{\partial \lambda} = -[P^\frac{1}{7} L^\frac{6}{7} - q] = 0
\]

The 1\textsuperscript{st} FOC implies:

\[
\frac{\partial \lambda}{\partial P} = -P_p - \frac{\lambda}{7} P^\frac{6}{7} L^\frac{6}{7} = 0
\]

\[
-\frac{\lambda}{7} P^\frac{6}{7} L^\frac{6}{7} = P_p
\]

\[
\lambda = -\frac{7P_p}{P^\frac{6}{7} L^\frac{6}{7}}
\]

The 2\textsuperscript{nd} FOC implies:
\[
\frac{\partial \mathcal{L}}{\partial L} = -P_L - \frac{6\lambda}{7} P^1 L^{-1} = 0
\]

\[
- \frac{6\lambda}{7} P^1 L^{-1} = P_L
\]

\[
\lambda = -\frac{7}{6} \frac{P_L}{P^1 L^{-1}}
\]

Equating the \(\lambda\)'s yields the familiar "the optimal input bundle is where the iso-quant is tangent to the iso-cost" result:

\[
- \frac{7P_p}{P^6 L^7} = -\frac{7}{6} \frac{P_L}{P^1 L^{-1}}
\]

\[
\frac{P_p}{P_L} = \frac{1}{6} \frac{P^6 L^7}{P^1 L^{-1}}
\]

\[
\frac{P_p}{P_L} = \frac{1}{6} \frac{L}{P}
\]

This allows us to isolate power (or for that matter labor) in terms of labor (power). For instance:

\[
\frac{P_p}{P_L} = \frac{1}{6} \frac{L}{P}
\]

\[
L = 6P \frac{P_p}{P_L}
\]

We can substitute this in the 3\textsuperscript{rd} FOC:

\[
\frac{\partial \mathcal{L}}{\partial \lambda} = - \left[ P^1 \frac{6}{7} - q \right] = 0
\]

\[
\frac{1}{7} \frac{6}{P^1 L^7} = q
\]

\[
P^7 \left( 6P \frac{P_p}{P_L} \right)^7 = q
\]
\[
\begin{align*}
\frac{1}{p_p} \left( 6 \frac{p_p}{p_L} \right)^6 &= q \\
\frac{1}{p^7} \left( 6 \frac{p_p}{p_L} \right)^6 &= q \\
P \left( 6 \frac{p_p}{p_L} \right)^6 &= q \\
P &= \frac{q}{\left( 6 \frac{p_p}{p_L} \right)^6}
\end{align*}
\]

Now, we are told that \( AVC_p = 4 \) which means that:

\[
TVC_p = P_p P = 4q
\]

Substitute the expression for \( P \) to get:

\[
TVC_p = P_p P = 4q
\]

\[
P_p P = 4q
\]

\[
P_p \left( \frac{q}{6 \left( \frac{p_p}{p_L} \right)^6} \right) = 4q
\]

\[
P_p \left( \frac{1}{6 \left( \frac{p_p}{p_L} \right)^6} \right) = 4
\]

\[
P_p \left( \frac{1}{6 \left( \frac{1}{p_L} \right)^7} \right) = 4
\]

\[
P_p = 4 \left( 6 \frac{1}{p_L} \right)^6
\]
\[ p_p = 4^2 \left( \frac{1}{6 \cdot \frac{1}{p_L}} \right)^6 \]

Substitute \( P_L = 30.25 \) to get:

\[ p_p = 4^2 \left( \frac{1}{6 \cdot \frac{1}{30.25}} \right)^6 \]

\[ p_p = 0.9976 \approx 1 \]
Question 2 [40 Points. Parts (d) & (e) worth 5 points each, all other parts worth 10 points each]

This question is based on the HBS case *The Aluminum Industry in 1994*. The following table contains the cost structure of the average CIS primary aluminum smelter, the average state primary aluminum smelter, and the average rational primary aluminum smelter (please note that cumulative capacity below is the total capacity of all smelters within a category (for example, the total cumulative capacity of all CIS smelters is 1.788 million tons per year):

<table>
<thead>
<tr>
<th>Smelter</th>
<th>Average CIS Smelter</th>
<th>Average State Smelter</th>
<th>Average Rational Smelter</th>
</tr>
</thead>
<tbody>
<tr>
<td>Country</td>
<td>CIS</td>
<td>All over</td>
<td>All over</td>
</tr>
<tr>
<td>Company</td>
<td>CIS</td>
<td>State</td>
<td>Rational</td>
</tr>
<tr>
<td>Average Capacity (‘000s tpy)</td>
<td>243.73</td>
<td>120.74</td>
<td>121.53</td>
</tr>
<tr>
<td>Total electricity cost</td>
<td>148.62</td>
<td>454.49</td>
<td>292.29</td>
</tr>
<tr>
<td>Total alumina cost</td>
<td>382.13</td>
<td>407.27</td>
<td>348.49</td>
</tr>
<tr>
<td>Other raw materials</td>
<td>63.69</td>
<td>163.57</td>
<td>120.62</td>
</tr>
<tr>
<td>Plant power and fuel</td>
<td>4.51</td>
<td>11.81</td>
<td>10.34</td>
</tr>
<tr>
<td>Consumables</td>
<td>76.92</td>
<td>56.72</td>
<td>73.91</td>
</tr>
<tr>
<td>Maintenance</td>
<td>39.57</td>
<td>46.45</td>
<td>53.84</td>
</tr>
<tr>
<td>Labor</td>
<td>17.80</td>
<td>62.73</td>
<td>194.19</td>
</tr>
<tr>
<td>Freight</td>
<td>68.76</td>
<td>53.17</td>
<td>37.82</td>
</tr>
<tr>
<td>General and administrative</td>
<td>67.11</td>
<td>52.48</td>
<td>86.58</td>
</tr>
<tr>
<td>Cumulative capacity (‘000s of tons/year) (all smelters in a category)</td>
<td>1,788.07 (All CIS)</td>
<td>2,826.95 (All state)</td>
<td>16,962.17 (All rational)</td>
</tr>
<tr>
<td>Total variable costs per ton ($/ton) = $VC(q)</td>
<td>740.14</td>
<td>1,135.25</td>
<td>873.15</td>
</tr>
</tbody>
</table>
(a) Solve the profit maximizing problem for the average rational primary aluminum smelter:

$$\max_q \Pi = PQ - TFC - TVC(q) \text{ s.t. } q \geq 0, q \leq q_c$$

Here $P = \text{price of aluminum/ton}$, $TFC = \text{Total fixed Cost}$, $TVC(q) = \text{Total variable cost and } q_c = \text{capacity}$. Assume $P > 0$. Show all calculations.

**Answer**

Since the case reports cost figures in “$ per ton” we assume that all smelter’s have constant returns so that:

$$TVC(q) = \text{constant } \cdot q$$

$$MC(q) = AVC(q) = \text{constant}$$

This will be useful below. Now, a smelter’s profit maximization problem (PMP) is:

$$\max_q \Pi = PQ - TFC - TVC(q) \text{ s.t. } q \geq 0, q \leq q_c$$

Note that smelters are price takers so that $P$ is a constant. Recall that all inequality constraints must be expressed in the form $g(x) \leq c$. Therefore:

$$\max_q \Pi = PQ - TFC - TVC(q) \text{ s.t. } q \geq 0, q \leq q_c$$

$$\max_q \Pi = PQ - TFC - TVC(q) \text{ s.t. } -q \leq q \leq q_c$$

Having expressed all constraints in terms of $g(x) \leq c$, form the Lagrangian:

$$\max_{q, \lambda_1, \lambda_2} \mathcal{L} = PQ - TFC - TVC(q) - \lambda_1 [q - q_c] - \lambda_2 [-q + 0]$$

$$\max_{q, \lambda_1, \lambda_2} \mathcal{L} = PQ - TFC - TVC(q) - \lambda_1 [q - q_c] + \lambda_2 q$$

The FOC is:

$$\frac{\partial \mathcal{L}}{\partial q} = P - \frac{dTVC(q)}{dq} - \lambda_1 + \lambda_2 = 0$$

The Kuhn-Tucker conditions are:

$$\lambda_1 \geq 0, q \leq q_c, \lambda_1 [q - q_c] = 0$$
\[ \lambda_2 \geq 0, q \geq 0, \lambda_2 q = 0 \]

Notice there are 4 possible cases that must be checked:

- **Case A**
  \[ q = 0, q = q_c \]
  Need to check if \( \lambda_2 \geq 0, \lambda_1 \geq 0 \).

  This requires that \( q = 0 \) and \( q = q_c > 0 \). Thus case A is impossible.

- **Case B**
  \[ q = 0, \lambda_1 = 0 \]
  Need to check if \( \lambda_2 \geq 0, q \leq q_c \).

  Since \( q = 0 \) the KT condition \( q \leq q_c \) is automatically satisfied. Thus we need to check if \( \lambda_2 \geq 0 \). Start with the FOC:

\[
\frac{\partial \mathcal{L}}{\partial q} = P - \frac{dTVC(q)}{dq} - \lambda_1 + \lambda_2 = 0
\]
\[ P - MC(q) - \lambda_1 + \lambda_2 = 0 \]

Substitute \( q = 0 \) and \( \lambda_1 = 0 \):

\[ P - MC(0) + \lambda_2 = 0 \]

\[ \lambda_2 = -P + MC(0) \]

Thus, for \( \lambda_2 \geq 0 \) we need:

\[ \lambda_2 = -P + \frac{dTVC(0)}{dq} \geq 0 \]

\[ P \leq MC(0) \]

\[ P \leq MC(0) \]

Case B is the solution when the price of aluminum is lower than \( MC(0) \). This is because the marginal cost of producing the 1st unit is greater than or equal to the price, or the \( MR \), of the 1st unit.

Thus, anytime \( P \leq MC(0) \) the competitive firm’s supply curve is \( q_s = 0 \) (that’s not the same as the MC curve):
Case B when $P \leq MC(0)$

<table>
<thead>
<tr>
<th>$q$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$-P + MC(0)$</td>
</tr>
</tbody>
</table>

Case C

$q = q_c, \lambda_2 = 0$

Need to check if $\lambda_1 \geq 0, q \geq 0$

Since $q = q_c > 0$ the KT condition $q \geq 0$ is automatically satisfied. Thus we need to check if $\lambda_1 \geq 0$:

$$\frac{\partial \xi}{\partial q} = P - \frac{dTVC(q)}{dq} - \lambda_1 + \lambda_2 = 0$$

$$P - MC(q) - \lambda_1 + \lambda_2 = 0$$

Substitute $q = q_c$ and $\lambda_2 = 0$:

$$P - MC(q_c) - \lambda_1 = 0$$

$$\lambda_1 = P - MC(q_c)$$

Thus, for $\lambda_1 \geq 0$ we need:
Case C will be the solution if the aluminum price, the $MR$, is greater than $MC(q_c) --$ the marginal cost at full capacity:

Thus, anytime $P \geq MC(q_c)$ the competitive firm’s supply curve is $q_s = q_c$. Again notice that the competitive firm’s MC curve is not the supply curve:
Constant Returns Technology

| Case C when \( P \geq MC(q_c) \) | \( q = q_c \) | \( \lambda_1 = P - MC(q_c) \) | \( \lambda_2 = 0 \) |

**Case D**

\( \lambda_1 = 1, \lambda_2 = 0 \)

Need to check if \( q \geq 0, q \leq q_c \)

Start with the FOC:

\[
\frac{\partial \xi}{\partial q} = P - \frac{dTVC(q)}{dq} - \lambda_1 + \lambda_2 = 0
\]

\[
P - MC(q) - \lambda_1 + \lambda_2 = 0
\]

Substitute \( \lambda_1 = \lambda_2 = 0 \)

\[
P - MC(q) = 0
\]

\[
P = MC(q)
\]
This is the familiar ECO 100 result that a competitive firm produces where price equals marginal cost. The only problem is that we don’t know when case D will be a solution for sure. For that we need to the conditions under which \( q \geq 0 \) and \( q \leq q_c \). From:

\[
P = MC(q)
\]

We have:

\[
q = MC^{-1}(P)
\]

Thus:

\[
0 \leq q \leq q_c
\]

\[
0 \leq MC^{-1}(P) \leq q_c
\]

\[
MC(0) \leq P \leq MC(q_c)
\]

This gives us a condition for case D to be a solution and for the output supplied to be between 0 and full capacity.

Intuitively, case D says that if the price of the product is between \( MC(0) \) and \( MC(q_c) \) the firm will produce an output between zero and full capacity:
Thus, anytime $MC(0) < P < MC(q_c)$ the competitive firm’s supply curve is also its MC curve:

![Constant Returns Technology](image)

<table>
<thead>
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<th>Case D when $MC(0) \leq P \leq MC(q_c)$</th>
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Putting all cases together we have a competitive smelter’s supply curve:
Put another way, the quantity supplied is:

\[ q_s = \begin{cases} 
0, & P < MC(0) \\ 
MC^{-1}(P), & MC(0) < P < MC(q_c) \\ 
q_c, & P \geq MC(q_c) 
\end{cases} \]

(b) Based on your answer to part (a) what is the impact on the average rational smelter’s optimal profits from, holding all else constant, a 1% increase in:

- The price of aluminum?
- Capacity?
- The minimum output?
- Fixed cost?

Assume that \( P = $1,100/\text{ton} \). Show all calculations.

**Answer:**
We are being asked to investigate the impact on the average rational smelter’s profits due to a change in a parameter – the easiest way to solve this is by the envelope theorem. To do this, we must first find out the average rational smelter’s output. Recall that:
All cost figures are $/t
Variable costs are in **BOLD**

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<td>1,135.25</td>
<td>873.15</td>
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Since we assumed all smelters have constant returns, the average **rational** smelter’s $MC(q) = $873.15 for $q = [0, q_c] = [0, 121.53]$:
Currently, the price of aluminum is $1,100/ton and since:

\[ P = 1,100 > 873.15 = MC(q) \]

The average rational smelter will produce at full capacity (i.e. case “C”):
Thus: $q^* = q_c = 121.53$ and:

$$\lambda_1^* = P - MC(q_c) = 1,100 - 873.25 = 226.75$$

$$\lambda_2^* = 0$$

$$\Pi^* = Pq - TFC - TVC(q)$$

Since $MC(q) = 873.25$ then $TVC(q) = 873.25q = 873.25(121.52) \approx 106,117$. Moreover, adding up all average fixed cost items we have:

$$AFC \approx 345 \rightarrow TFC = AFC \cdot q = 345(121.53) \approx 41,928$$

Therefore:

$$\Pi^* = Pq - TFC - TVC(q)$$

$$\Pi^* = (1,100)(121.52) - 41,928 - 106,117$$

$$\Pi^* = 133,672 - 41,928 - 106,117 = -14,373$$

Notice that producing at full capacity maximizes profits in the sense that it minimizes loss. (As a study question, you should check that operating at full capacity is better than shutting down, confirming the rule that a rational company incurring losses should shut down when $P < AVC(q)$).

Now, what is the impact on the average rational smelter’s optimal profits from, holding all else constant, a 1% increase in:

- The price of aluminum?
- Capacity?
- The minimum output?
- Fixed cost?

By the envelope theorem, the change in the objective (in this case profits) from a small change in a parameter is gotten by differentiating the Lagrangian with respect to the parameter, evaluated at the initial solution.

Optional note: Recall the Lagrangian was:
\[
\max_{q, \lambda_1, \lambda_2} \mathcal{J} = Pq - TFC - TVC(q) - \lambda_1[q - q_c] - \lambda_2[-q - 0]
\]

Any solution to this problem must satisfy the KT conditions:

\[
\lambda_1 \geq 0, q \leq q_c, \lambda_1[q - q_c] = 0
\]

\[
\lambda_2 \geq 0, q \geq 0, \lambda_2 q = 0
\]

The “product” terms in the KT conditions imply that at the optimum the following terms are zero:

\[
\max_{q, \lambda_1, \lambda_2} \mathcal{J} = Pq - TFC - TVC(q) - \lambda_1[q - q_c] - \lambda_2[-q - 0]
\]

This is why differentiating the optimal Lagrangian is equivalent to differentiating optimal profits with respect to the parameter.

---

Envelope Theorem: the marginal profit due to a 1\% increase in aluminum price

The Lagrangian was:

\[
\max_{q, \lambda_1, \lambda_2} \mathcal{J} = Pq - TFC - TVC(q) - \lambda_1[q - q_c] - \lambda_2[-q + 0]
\]

Differentiating the Lagrangian with respect to aluminum price:

\[
\frac{\Delta \Pi}{\Delta P} = \frac{\partial \mathcal{J}}{\partial P} = q
\]

This is the impact on profits from a $1 increase in aluminum price. To find the impact due to a 1\% increase in aluminum price we have:

\[
\frac{\% \Delta \Pi}{\% \Delta P} = \frac{\partial \mathcal{J}}{\partial P} \frac{P}{\mathcal{J}} = q \frac{P}{\mathcal{J}} = q \frac{P}{\Pi}
\]

Evaluate at optimal solution:

\[
\frac{\% \Delta \Pi}{\% \Delta P} = q^* \frac{P^*}{\Pi^*}
\]

\[
\frac{\% \Delta \Pi}{\% \Delta P} = \frac{(121.52)(1,100)}{-14,373} = -9.3\%
\]

\[
\% \Delta \Pi = -9.3\% (\% \Delta P)
\]

\[
\% \Delta \Pi = -9.3\% (+1 \%)
\]
\[
\% \Delta \Pi = -9.3\%
\]

\[
\frac{\text{New } \Pi - \text{Initial } \Pi}{\text{Initial } \Pi} \times 100 = -9.3\%
\]

\[
\frac{\text{New } \Pi - \text{Initial } \Pi}{\text{Initial } \Pi} = -0.093
\]

\[
\text{New } \Pi - \text{Initial } \Pi = -0.093 \text{ Initial } \Pi
\]

\[
\text{New } \Pi = \text{ Initial } \Pi - 0.093 \text{ Initial } \Pi
\]

\[
\text{New } \Pi = -14,373 - 0.093(-14,373)
\]

\[
\text{New } \Pi = -13,036
\]

A 1% increase in aluminum prices reduces the average rational smelter’s loss by 9.3% or from a loss of ($14,373) to a loss of ($13,036).

**Envelope Theorem: the marginal profit due to expanding capacity by 1%**

The Lagrangian was:

\[
\max_{q, \lambda_1, \lambda_2} \mathcal{L} = Pq - TFC - TVC(q) - \lambda_1[q - q_c] - \lambda_2[-q + 0]
\]

Differentiating the Lagrangian with respect to capacity:

\[
\frac{\Delta \mathcal{L}}{\Delta q_c} = \frac{\partial \mathcal{L}}{\partial q_c} = \lambda_1
\]

This is the impact on profits from a 1 unit increase in capacity. To find the impact due to a 1% increase in capacity we have:

\[
\frac{\% \Delta \Pi}{\% \Delta q_c} = \frac{\partial \mathcal{L}}{\partial q_c} \frac{q_c}{\mathcal{L}} = \lambda_1 \frac{q_c}{\mathcal{L}} = \lambda_1 \frac{q_c}{\Pi}
\]

Evaluate at optimal solution:

\[
\frac{\% \Delta \Pi}{\% \Delta q_c} = \lambda_1^* \frac{q_c}{\Pi^*}
\]

\[
\frac{\% \Delta \Pi}{\% \Delta q_c} = 344.95 \frac{121.52}{-14,373} = -2.92\%
\]

\[
\% \Delta \Pi = -2.92\% (+1\%)
\]

\[
\% \Delta \Pi = -2.92\%
\]
\[
\text{New \(\Pi\) } - \text{Initial \(\Pi\)} \quad \frac{\text{New \(\Pi\) } - \text{Initial \(\Pi\)}}{\text{Initial \(\Pi\)}\times 100 = -2.92\% \\
\text{New \(\Pi\) } - \text{Initial \(\Pi\)} \quad \frac{\text{New \(\Pi\) } - \text{Initial \(\Pi\)}}{\text{Initial \(\Pi\)} = -0.0292 \\
\text{New \(\Pi\) } = \text{Initial \(\Pi\)} - 0.0292 \text{ Initial \(\Pi\) } \\
\text{New \(\Pi\) } = -14,373 - 0.0292 (-14,373) \\
\text{New \(\Pi\) } = -13,953.31
\]

A 1\% increase in capacity reduces the average rational smelter’s loss by 2.92\% or from a loss of ($14,373) to a loss of ($13,953.31).

**Envelope Theorem:** the marginal profit due to raising the minimum output by 1\%

The Lagrangian was:

\[
\max_{q,\lambda_1,\lambda_2} \mathcal{L} = Pq - TFC - TVC(q) - \lambda_1[q - q_c] - \lambda_2[-q + 0]
\]

We had required that \(q \geq 0\). Re-writing this constraint as \(q \geq q_{\text{min}}\) we have:

\[
\max_{q,\lambda_1,\lambda_2} \mathcal{L} = Pq - TFC - TVC(q) - \lambda_1[q - q_c] - \lambda_2[-q + q_{\text{min}}]
\]

Differentiating the Lagrangian with respect to \(q_{\text{min}}\):

\[
\frac{\Delta \Pi}{\Delta q_{\text{min}}} = \frac{\partial \mathcal{L}}{\partial q_{\text{min}}} = -\lambda_2
\]

This is the impact on profits from a 1 unit increase in minimum output. To find the impact due to a 1\% increase in minimum output we have:

\[
\frac{\% \Delta \Pi}{\% \Delta q_{\text{min}}} = \frac{\partial \mathcal{L}}{\partial q_{\text{min}}} \frac{q_{\text{min}}}{\mathcal{L}} = \frac{q_{\text{min}}}{\mathcal{L}} \Rightarrow \frac{\% \Delta \Pi}{\% \Delta q_{\text{min}}} = -\lambda_2 \frac{q_{\text{min}}}{\mathcal{L}} = -0 \frac{q_{\text{min}}}{\mathcal{L}} = 0
\]

There is no impact on optimal profits from raising the minimum output requirement by 1\%. Why? Because the optimal solution is for the smelter to produce well above zero, so that the minimum output constraint does not bind – as such, there is no value in relaxing the constraint.

**Envelope Theorem:** the marginal profit of 1\% higher fixed cost

The Lagrangian was:
\[ \max_{\lambda_1, \lambda_2} J = Pq - TFC - TVC(q) - \lambda_1[q - q_c] - \lambda_2[-q + 0] \]

Differentiating the Lagrangian with respect to \( TFC \):

\[ \frac{\Delta \Pi}{\Delta TFC} = \frac{\partial J}{\partial TFC} = -1 \]

This is the impact on profits from a $1 increase in TFC. To find the impact due to a 1% increase in TFC we have:

\[ \frac{\% \Delta \Pi}{\% \Delta TFC} = \frac{\partial J}{\partial TFC} \frac{TFC}{J} = -1 \frac{TFC}{J} = -1 \frac{TFC}{\Pi} \]

Evaluate at optimal solution:

\[ \frac{\% \Delta \Pi}{\% \Delta TFC} = -1 \frac{TFC}{\Pi^*} \]

\[ \frac{\% \Delta \Pi}{\% \Delta TFC} = -1 \frac{41,928}{-14,373} = +2.92\% \]

\[ \% \Delta \Pi = + 2.92\% (+ 1 \%) \]

\[ \% \Delta \Pi = + 2.92\% \]

\[ \frac{\text{New } \Pi - \text{Initial } \Pi}{\text{Initial } \Pi} * 100 = + 2.92\% \]

\[ \frac{\text{New } \Pi - \text{Initial } \Pi}{\text{Initial } \Pi} = + 0.0292 \]

\[ \text{New } \Pi - \text{Initial } \Pi = + 0.0292 \text{ Initial } \Pi \]

\[ \text{New } \Pi = \text{Initial } \Pi + 0.0292 \text{ Initial } \Pi \]

\[ \text{New } \Pi = -14,373 + 0.0292 (-14,373) \]

\[ \text{New } \Pi = -14,793 \]

A 1% increase in TFC raises the average rational smelter’s loss by 2.92% or from a loss of ($14,373) to a loss of ($14,793).
(c) Graph the primary aluminum industry supply curve as if all CIS smelters behave like the average CIS smelter, all state owned smelters behave like the average state smelter, and all rational smelters behave like the average rational smelter.

**Answer:**

From the case we know that CIS and state owned smelters most likely behave as irrational smelter’s, i.e. they produce aluminum even if the price falls below $AVC$. Put another way, irrational smelters produce aluminum regardless of the price. Recall that:

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Assuming constant returns and that smelters produce at full capacity (for cases C and D) we see that together, the state and CIS smelters will produce a cumulative output of 1,788.07 + 2,826.95 = 4,615.02 ‘000s tons at any price while the rational smelters will produce 16,962.17 ‘000s of tons so long as \( P \geq MC(q) = 873.15 \):

(d) [This part is independent of all parts below] Graph the primary aluminum industry supply curve from part (c) below and then show the impact of all rational smelters experiencing “learning by doing” (assume learning by doing has a small effect).

**Answer**

With learning by doing there will be a decrease in every smelters’ \( AVC \). However, since the state and CIS smelters produce at any price, there will be no change in the their “supply” curve whilst learning by doing pushes the rational smelters’ supply curve down:
(e) Use the primary aluminum supply model in part (c) to predict the demand for primary aluminum for the case for the cases when (i) $P = 800/\text{ton}$, (ii) $P = 1100/\text{ton}$. Please show all graphs below and explain your reasoning.

**Answer:**
Case (i): $P = 800/\text{ton}$. In this case:
Notice that all CIS and state smelters will produce aluminum while all rational smelters will shut down. The total output will be 4.61m tons. Assuming no inventory buildups or rundown, demand for primary aluminum will be 4.61m tons.

Case (ii): $P = \$1,100/\text{ton}$. In this case:
Notice that all smelters will produce aluminum. The total output will be 21.588m tons. Assuming no inventory buildups or rundown, demand for primary aluminum will be 21.588m tons.

The End
😊

WORKSHEETS