Structural Estimation of Auction Models

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In recent years there has been much empirical work using auction data. One clear reason for this is that auctions are being increasingly used as real-world allocation mechanisms (the most celebrated example being the spectrum auctions run the the U.S. Federal Communications Commission), thereby raising many interesting empirical issues. On the other hand, auction theory is among the more systematically developed branches of economics with many clear-cut predictions which are ripe for testing. This combination of highly-developed theory and real-world applications has spawned an impressive body of empirical work.

This paper examines one aspect of this empirical work — that dealing with structural estimation of auction models. This work (see Paarsch (1992), Paarsch (1991) for two of the earliest examples) derives the estimating equations directly from the equilibrium bid functions posited in the theoretical auction literature, and attempts to recover the parameters of the underlying distribution of bidders' valuations. In contrast, the *reduced-form* empirical auction literature (see Hendricks and Porter (1988) for an example) tests the comparative statics predictions of the theoretical auction models, without directly recovering the parameters of the distribution of bidders' valuations.

These two approaches are used to address different types of questions. The reduced-form approach, which aims more to characterize bidder behavior in an auction rather than to use the equilibrium bid functions as a mapping from observed bids to (unobserved) bidder valuations in order to estimate the parameters of the distribution of the latter. For example, the analysis of Hendricks and Porter (1988) tries to uncover patterns between the observed bids, the number of participating bidders, and (proxies of) differences in informedness among bidders in offshore oil and gas auctions, patterns which are predicted by auction theory.

On the other hand, the structural approach explicitly recovers the parameters of the distribution of bidders' valuations. These parameters allow the researcher to simulate auction results under alternative auction formats, which is crucial for comparing the efficiency and seller revenue optimality of alternative auction forms, as well as evaluating the effects of policy changes. For example, Paarsch (1991) uses his estimates to calculate the optimal reserve prices for British Columbian timber auctions, which are an important source of governemnt revenue in regions in Western Canada. The price paid for the structural approach is the extra assumptions that the researcher must make, relatively to the reduced-form approach. In short, the researcher must make the same assumptions that are made in the theoretical models. By definition, the estimating equation in a structural auction model is derived from the equilibrium bid functions of a theoretical auction model. Structural estimation is (again by definition) impossible for auction formats for which the forms of the equilibrium bid functions are not known.

This paper offers a general view of structural estimation using auction data, and emphasizes the common components behind most structural econometric auction model. In the next section, we discuss the general framework of a structural auction model. Section 2 illustrates this framework with several examples from the literature. Section 3 discusses estimation methodologies and the econometric problems encountered in the estimation procedure. Section 4 concludes.

1 A general structural empirical auction model

We restrict our attention to single object auctions. An auction has N bidders (indexed $i = 1, \ldots, N$), each of whom have a valuation V_i for the object, and receive a private signal X_i about V_i. Bidder i only observes X_i prior to the beginning of the auction. He doesn't observe any of the valuations, V_j, for $j = 1, \ldots, N$, or any of the other bidder's signal, X_j, for $j \neq i$.

The bidders' valuation s and private signals are jointly distributed according to the distribution function $F(V_1, \ldots, V_N, X_1, \ldots, X_N)$. The researcher estimates a structural model in order to identify the F distribution, or parameters thereof.

An auction model is distinguished by (i) the *form* of the auction being studied and (ii) the assumptions underlying the joint distribution of the bidders' valuations and their signals, which we call the *paradigm* of the auction.

The form of an auction includes the bidding rules (e.g. sealed-bid vs. open-cry, whether there is a reserve price) as well as the allocation rule which dictates what price the winning bidder must pay for the object. The most commonly used auction forms are first-price auctions, second-price auctions, Dutch auctions and English(ascending) auctions. Particulars about these auction forms are given in Milgrom and Weber (1982).

We refer to the set of assumptions made regarding $F(V_1, \ldots, V_N, X_1, \ldots, X_N)$ as the "paradigm" of a particular auction model. In the pure *private value* paradigm, $V_i = X_i \ \forall \ i$ (i.e. each bidder knows his true valuation for the object) while in the pure *common value* paradigm $V_i = V, \ \forall \ i$ (i.e. the value of the object is the same to all bidders, but none of the bidders knows the true value of the object; here the individual X_i's are noisy signals of the true but unknown V). Typically, however, we would expect that there are both private and common value components in the valuation that a bidder places on the object on sale. Generally speaking, V_i is a function of *all* the bidders' signals: and other information variables that may not be observed by any of the bidders.

When the joint distribution F is symmetric with respect to $1, \ldots, N$, the model is *symmetric*. Otherwise it is *asymmetric*. When the joint distribution F can be factored into marginal distributions of V_i, X_i, $i = 1, \ldots, N$ (i.e., $F(V_1, \ldots, V_N, X_1, \ldots, X_N) = F_1(V_1, X_1)\ F_2(V_2, X_2)\ \ldots\ F_N(V_N, X_N)$), the bidders' valuations are labelled *independent*. Different combinations of private/common value assumptions, symmetric/asymmetric distribution assumptions and independence/dependence assumptions create a host of possible paradigms for each auction form. Milgrom and Weber (1982) provided the seminal analysis for the symmetric versions of most of the usual auction forms; much of the recent theoretical work has focused on asymmetric cases (Maskin and Riley (1996), Bulow, Huang, and Klemperer (1996)).

The data are the observed bids $p_1, \ldots, p_N$. In many applications the researcher only observes the winning bid. In other applications, the researcher observes all the bids, but may choose only to estimate using winning bid data.

Given the assumptions underlying each model paradigm, a structural empirical auction model has two components:

1. Equilibrium bid functions Theoretical equilibrium characterization results contribute the form that the equilibrium bidding strategies take. Bidder i's equilibrium bid is a function of all the information variable available to him, denoted by Ω_i. In a sealed bid auction, the only informtion variable he has is his own private signals, X_i. On the other hand, in an irreversible dropout ascending auction[1], the information variables available to him during a stage of the auction include all the signals of the already dropped-out bidders, which bidder i infers from the dropped-out prices.

Generally, bidder i's equilibrium bid can be considered a function of all the private signals: $b_i(X_1, \ldots, X_N)$,

[1] the form considered in Milgrom and Weber (1982) and in most subsequent work on ascending auctions

with the understanding that the form of the bid function $b_i(\cdots)$ depends both on the auction form, and on F, the joint distribution of valuation and signals. For example, in seal bid auctions, $b_i(\cdots)$ depends only on X_i. The collection of equilibrium bid functions $b_1(\cdots;F), \ldots, b_N(\cdots;F)$ provide a mapping between the private signals $X_1, \ldots, X_N$ and the observed bids $p_1, \ldots, p_N$. Monotonicity assumptions about the equilibrium bid functions usually ensure that this mapping is one-to-one.

2. Researcher's distribution assumptions regarding F Given this mapping, the distributional of the signals X's induce a joint distribution function for the observed bids $G(p_1, \ldots, p_N)$, usually in the form of a joint density function $g(p_1, \ldots, p_N)$, which forms the basis for an estimating equation. An interesting feature of the structural auction model is that $G(p_1, \ldots, p_N)$ depends on F in two ways.

Consider $g(p_1, \ldots, p_N) = f_X(X_1, \ldots, X_N) \times J(X_1, \ldots, X_N | p_1, \ldots, p_N)$, where $J(\cdots)$ denotes the Jacobian transformation from the private signals to the observed prices. $f_X(\cdots)$ is simply given by the marginal joint density of $X_1, \ldots, X_N$ from F. On the other hand, $J(\cdots)$ usually also depends on F, in the sense that for the same auction form, j will vary depending on assumptions made about F.

This latter point contrasts with traditional sample selection or limited dependent variable models, in which the Jacobians of the mappings from the unobserved continuous variables to the observed (partially-)discrete variables do not vary depending on assumptions made about the distribution of the unobserved variables.[2] This analogy between auction models and latent variable models raises questions about identification and estimation which have been addressed in the sample selection literature (cf. Manski (1995)). A recent series of papers, e.g. Vuong, Perrigne, and Li (1997) and Vuong, Perrigne, and Guerre (1996) have addressed the issue of nonparametric identification in auction models, and developed nonparametric estimation techniques.

The joint distribution of observed bid data, $G(p_1, \ldots, p_N; F)$ provides a way of identifying the underlying latent distribution F (which is the ultimate goal of structural estimation of auction models) from the observed bids. In the following we will focus on parametric models, in which the form of F is assumed to be known up to a finite dimensional vector of parameters. All the information needed for any estimation procedure is summerized in $g(\cdot)$, the joint density of the bids. In maximum likelihood estimation procedure, we use $g(\cdots)$ directly as the estimating equation. This is the approach taken by in the Paarsch papers cited earlier. Alternatively, the moments of g can also be used to do minimum distance estimation (nonlinear least squares), as in Laffont,

[2] For example, the Jacobian for the mapping between the unobserved utility levels and the observed discrete choice indicators in a binary choice model is invariant to whether the model is a binary logit, probit, or any alternative.

Ossard, and Vuong (1995) and Hong and Shum (1997). In what follows, we present a survey of previous work on structural estimation of auction models, fitting these examples into the general framework described above.

2 Deriving distribution of observed bid data: some examples

2.1 First-price auction models

Laffont, Ossard, and Vuong (1995) considered symmetric independent private value (IPV) models, which imply that $F_X(x_1,\ldots,x_N)=\prod_{i=1}^{N} H(x_i)$. In these models, the equilibrium bid functions are defined by a first-order differential equation

$$b'(x)F(x)^N + Nb(x)f(x)F(x)^{N-1} = Nxf(x)F(x)^{N-1}$$

together with the boundarty condition that $b(p_0)=p_0$. Given the symmetry assumption, the bid function is expressible analytically as:

$$b(X_i) = X_i - \frac{1}{(H(X_i))^{N-1}}\int_{p_0}^{X_i} (H(x))^{N-1}dx \tag{1}$$

where p_0 is the reservation price in the auction.

Without simplifying parametric assumptions, no closed-form solution exists for g in this case, given the nonlinearity in the transformation from x_i to p_i expressed in equation (1). However, for the symmetric IPV framework, Laffont, Ossard, and Vuong (1995) invoke the revenue equivalence theorem which states that for the private value paradigm considered here, the expected revenue from a first- and second-price auction would be equivalent. Since they only observe the winning bid for their auctions, this theorem ensures that, in equilibrium, the winning bid will have the same expectation as the second-highest draw out of N draws from the H distribution, with corresponding density function given above in equation 4. In their simulated nonlinear least squares framework, they use this theorem to avoid having to simulate the equilibrium bid function (in equation 1) for any number of given draws of $(X_1,\ldots,X_N)$.

It is difficult to relax the symmetry and independence assumptions in first price auction models. The equilibrium bid functions in an asymmetric independent first price auction are defined by a system of first-order differential equations:

$$(X - b_i(X))\sum_{j\neq i}\frac{h_j\left(b_j^{-1}(b_i(X))\right)}{H_j\left(b_j^{-1}(b_i(X))\right)}\frac{1}{b_j'\left(b_j^{-1}(X)\right)} = 1, \quad \text{for } i=1,\ldots,N \tag{2}$$

with the boundary conditions $b_i(\underline{x}) = \underline{x}$, where $\underline{x}$ is the lower bound of the support of a signal X. Given the independence assumption, the joint density of the observed bids, which applies both to symmetric and asymmetric models, is therefore: $g(p_1, \ldots, p_N) = \prod_{i=1}^{N} \frac{h_i\left(b_i^{-1}(p_i)\right)}{b_i'\left(b_i^{-1}(p)\right)}$

Bajari (1996) has turned to computational procedures to solve the system of differential equations in (2). In a nonparametric framework, Vuong, Perrigne, and Guerre (1996) showed that it is possible to nonparametrically identify F from all bid data for private value first-price auction models, regardless of the asymmetry and dependence of F, as long as F is such that it gives rise to a strictly increasing strategy equilibrum.

2.2 Symmetric independent private value second-price ("Vickrey") auction

This is the case considered by Paarsch (1991). Here $F(X_1, \ldots, X_N) = \prod_{i=1}^{N} H(x_i)$, and $b_i = X_i$, $\forall i$ is unique increasing dominant-strategy equilibrium. Given these strategies, the joint density of observed bids is the same as (assumed) joint density of the bidder signals:

$$g(p_1, \ldots, p_N) = h(p_1)h(p_2)\ldots h(p_N) \tag{3}$$

Paarsch assumes that H is the Weibull distribution, a flexible two-parameter distribution which can have a monotonically increasing or decreasing hazard rate depending on the values of the parameters.

Given the simple form of the equilibrium bidding strategies in the second-price private value auction, if only the winning bid is observed, its density would be that of the second highest draw out of N draws from the H distribution:

$$f_{X(2:N)}(X) = N(N-1)[H(X)][1 - H(X)]^{N-2}h(X) \tag{4}$$

2.3 Asymmetric Open Auction Models

2.3.1 Private value models

Ever since Vickrey (1962), it has been known that in private value second-price auctions and ascending auctions (which we will jointly refer to as **open auctions**), bidding up to the private value ($b_i(X_i) = X_i$) is the unique weakly undominated strategy equilibrium, *regardless of symmetry or independence assumptions.*[3] Therefore, the Vickrey auction provides an example of an auction

[3] In fact, Vickrey generalizes this "highest rejected bid" principle to the simultaneous auctions of identical objects.

model where the Jacobian of the transformation from the unobserved X's to the observed p's does not explicitly depend on F, the joint distribution function for the X's. In equilibrium, $p_i = X_i$, so the Jacobian is simply the identity matrix. This property disappears once common value components are added to the model. This is the focus of the next section.

2.3.2 Models with common values

In common value auction models, it is assumed that there is an (unknown to all bidders) component in bidders' valuations of the object which is the same (i.e., "common") across all bidders. For this reason, each private signal signal X_i is useful to each bidder $j \neq i$ in estimating his valuation V_j, so that V_j is typically a function of *all* the private signals $X_1, \ldots, X_N$.

Note the subtle difference between a common value model and a model with correlated private values (such as that considered in Vuong, Perrigne, and Li (1997)). Both models assume correlation among the private signals, i.e., $F_X(X_1, \ldots, X_N)$ cannot be factored into $\prod_i H_i(X_i)$. In the correlated private value model, it remains the case that $V_i = X_i$, for $i = 1, \ldots, N$. In the common value model, however, $V_i = v_i(X_1, \ldots, X_N)$, for $i = 1, \ldots, N$, which differs from the private value case.

As we will show, equilibria in open auctions with common value components under both asymmetry and dependence have a very intuitive derivation, even allowing for asymmetry and dependence in bidders' private signals. One striking finding is that the *inverse* bid functions, i.e., what a bidder's signal would be if he chooses to bid the given price level p, can be derived as solutions to a system of nonlinear equations defined as bidders' expected valuations conditional on all the information available during that stage of the auction. From a computational point of view, this provides a way to derive the mapping from the unobserved X's to the observed p's numerically in the usual case when it is analytically intractable.

Here let us introduce the shorthand notation that bidder i's bid function be written simply as a function of his signal x_i and the public information set Ω: $b_i(x_i) \equiv b_i(x_i; \Omega)$. Then in these open auctions:

$$\begin{aligned} b_1(x_1) &= E[V_1 \mid \mathcal{A}_1(x_1, b_2(X_2), \ldots, b_N(X_N)), \Omega] \\ &\qquad \ldots \\ b_N(x_N) &= E[V_N \mid \mathcal{A}_N(b_1(X_1), \ldots, B_{N-1}(X_{N-1}), x_N), \Omega] \end{aligned} \tag{5}$$

where $\mathcal{A}_i(\cdots)$ denotes the conditioning event for bidder i. Typically, $\mathcal{A}_i$ involves bidder i's private signal X_i as well as his equilibrum beliefs about the other bidders' signal $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_N$. Furthermore, as will be seen below, $\mathcal{A}_i$ relates all the other bidders' signals to bidder i's signal through the equilibrium bid functions $b_i(X_i), i = 1, \ldots, N$. In what follows we will explicitly write out $\mathcal{A}_i$ for the second price and ascending auctions.

For a generic bid $p = b_i(x_i), i = 1, \ldots, N$, the special form of A_i allows us to rewrite the above system of conditional expectations as a system of N nonlinear equations, with N unknowns $\phi_1(p), \ldots, \phi_N(p)$, where $\phi_i(p) \equiv b_i^{-1}(p)$, the inverse bid function for bidder i evaluated at the generic bid p; i.e. the system (5) can be rewritten as

$$\begin{aligned} p &= E[V_1 \mid \mathcal{A}_1(\phi_1(p), \ldots, \phi_N(p)); \Omega] \\ &\vdots \\ p &= E[V_N \mid \mathcal{A}_N(\phi_1(p), \ldots, \phi_N(p)); \Omega]. \end{aligned} \tag{6}$$

By taking different values for p, we can solve the system (6) for the N inverse bid functions in pointwise fashion. Given any distribution F then, the existence, uniqueness, and monotonicity properties of the equilibrium bid functions can be directly verified from the existence, uniqueness, and monotonicity of solutions to the system of equations (6).

Note that assumptions about F, the joint distribution of bidders' private signals, determine in practice only the form that the conditional expectations will take. The form that the equilibrium bid functions take depends on the nature of the solution to the nonlinear system of equations posed by the conditional expectation equations. It is in this sense that J, the Jacobian of the mapping from the unobserved X's to the observed p's depends explicitly on the assumptions made regarding F. This is not the case for limited dependent variable (LDV) models where the mapping from unobserved utility indices to the observed LDV's are threshold-crossing conditions which are invariant to the distribution assumed for the unobserved utility indices.

Even given parametric assumptions about F, it is rare to find cases where the conditional expectations will have a closed form, much less cases where both the conditional expectations and the equilibrium bid functions are expresible in closed form. An example of this is the log-normal irreversible-dropout English auction model, examined by Wilson (1995) and recently implemented empirically by Hong and Shum (1997).

In general, solving for the equilibrium bid functions will require numerical procedures both at the conditional expectation evaluation stage (requring numerical integration) and the stage of solving for the implicitly-defined equilibrium bid functions (function approximation routines).

2.3.3 Asymmetric second price auctions

For general asymmetric second-price auctions, the equilibrium bidding strategies consist of one bid function per bidder, i.e. a set of functions $b_i(X_i)$, for $i = 1, \ldots, N$. In equilibrium, bidder i believes that his bid is equal to the highest competing bid, i.e.,$\max_{j \neq i} b_j(X_j) = b_i(X_i)$. Therefore, the equilibrium bid functions satisfy the following system of conditional expectations:

$$\begin{aligned} b_1(X_1) &= E[V_1 \mid X_1, \max_{j\neq 1} b_j(X_j) = b_1(X_1)] \\ b_2(X_2) &= E[V_2 \mid X_2, \max_{j\neq 2} b_j(X_j) = b_2(X_2)] \\ &\cdots \\ b_N(X_N) &= E[V_N \mid X_N, \max_{j\neq N} b_j(X_j) = b_N(X_N)]. \end{aligned} \tag{7}$$

Next we will rewrite these equations using the inverse bid functions, analogous to the system (6) above.

Define $\beta_{-i} \equiv \max_{j\neq i} b_j(X_j)$ and the function $\tilde{\mathcal{V}}_i(x,p) \equiv E(V_i \mid X_i = x, \beta_{-i} = p)$. We will first discuss the event $\{\beta_{-i} = p\}$. $\{\beta_{-i} = p\}$ means that the highest bid among bidders $j, j \neq i$ is p, which in turn implies that

§1 All bids by bidders $j, j \neq i$ are smaller than or equal to p. $b_j(X_j) \leq p, \ \forall\ j \neq i$.

§2 At least one of $b_j(X_j) = p, j \neq i$.

Therefore we can write the event $\{\beta_{-i} = p\}$ as

$$\left[\bigcap_{j\neq i}(b_j(X_j) \leq p)\right] \cap \left[\bigcup_{j\neq i} b_j(X_j) = p\right] = \left[\bigcap_{j\neq i}(X_j) \leq \phi_j(p)\right] \cap \left[\bigcup_{j\neq i}(X_j) = \phi_j(p)\right] \tag{8}$$

where $\phi_j(p) = b_j^{-1}(p)$ are the inverse bid functions of bidders $j, j \neq i$.[4] In view of (8), we can rewrite (7) as:

$$\begin{aligned} p &= E\left(V_1 \middle| X_1 = \phi_1(p), \left[\bigcap_{j\neq 1}(X_j) \leq \phi_j(p)\right] \cap \left[\bigcup_{j\neq 1}(X_j) = \phi_j(p)\right]\right) \\ &\vdots \\ p &= E\left(V_N \middle| X_N = \phi_N(p), \left[\bigcap_{j\neq N}(X_j) \leq \phi_j(p)\right] \cap \left[\bigcup_{j\neq N}(X_j) = \phi_j(p)\right]\right) \end{aligned} \tag{9}$$

[4] Because we assume that the type space is continuous, the event (8) has zero probability. However, the conditional expectation we are computing is well defined as long as we assume $f(v, x) > 0$ on its support.

which, analogously to the system (6), is a system of N equations in the N unknowns $\phi_1(p), \ldots, \phi_N(p)$ which can be solved for different p's for the inverse bid functions $\phi_1, \ldots, \phi_N$. [5]

2.3.4 Asymmetric ascending auctions

The ascending auction proceeds in rounds. It enters a new round whenever another bidder drops out. N bidders are present in the auction; there will be $N-1$ "rounds" in the auction, indexed $k = 0, \ldots, N-2$. In round 0, all N bidders are active; in round k, only $N-k$ bidders are active. Each round ends when a bidder drops out; bidders are indexed by $i = 1, \ldots, N$. Without loss of generality, the ordering $1, \ldots, N$ indicates the order of dropout. In other words, bidder N drops out in round 0, and bidder 1 wins the auction; generally, bidder $N-k$ drops out at the end of round k. The dropout prices are indexed by rounds, i.e. $P_0, \ldots, P_{N-2}$.[6] To sum up, bidder i drops out at the end of round $N-i$, at the price P_{N-i}.

Equilibrium bidding strategies in the ascending auction game specify, for each bidder i, bid functions $b_i^k(X_i)$ for each round k, $k = 0, \ldots, N-2$, i.e. $b_i^0(X_i), \ldots, b_i^{N-2}(X_i)$. Given a realization of the private signal X_i, the bid function $b_i^k(X_i)$ tells bidder i which price he should drop out at during round k. The collections of bid functions $b_i^0(X_i), \ldots, b_i^{N-2}(X_i)$ for bidders $i = 1, \ldots, N$ are common knowledge. The equilibrium conjectures A_i and the bidders' expectations (5) evolve during different rounds of the auction.

Again, consider bidder i, who is active during round k. As of round k, bidders $N-k+1, \ldots, N$

[5] In calculating the conditional expectations in (9), we will express it as:

$$\breve{\mathcal{V}}_i(x_1, \ldots, x_N) = E\left(V_i \middle| X_i = x_i, \bigcup_{j \neq i} [X_j = x_j, X_k < x_k, k \neq j, i]\right)$$
$$= \frac{\int V \sum_{j \neq i} \int^{x^1_{-ij}} \cdots \int^{x^{n-2}_{-ij}} f_{V_i, X_1, \ldots, X_N}\left(V, x_i, x_j, X^1_{-ij}, \ldots, X^{N-2}_{-ij}\right) dX^1_{-ij} \cdots dX^{N-2}_{-ij} dV}{\sum_{j \neq i} \int^{x^1_{-ij}} \cdots \int^{x^{N-2}_{-ij}} f_{X_1, \ldots, X_N}\left(x_i, x_j, X^1_{-ij}, \ldots, X^{N-2}_{-ij}\right) dX^1_{-ij} \cdots dX^{N-2}_{-ij}} \tag{10}$$

where X_{-ij} denotes the $N-2$ vector of private signals for bidders other than i and j, and X^k_{-ij} denotes the generic kth element of the vector X_{-ij}. Each term in the summation presents one event in the union; they are disjointed from each other. The $N-2$ integral in the denominator is the joint density of the conditioning event. It integrates over the signal of each of the bidders, other than i and j, from the lower bound $\underline{X}$ up to each of the $x_k = \phi_k(b)$. Unless the integrals can be analytically expressed, in general it can be computationally intensive to calculate this conditional expectation because it involves multi-dimensional integrals.

In particular, even for jointly normally distributed (v, x), the difficulty of evaluating the multivariate normal distribution function is similar to that encountered in the estimation of multivariate probit models. One recently proposed solution of this problem is to use a simulation estimation, which evaluates the integral by the empirical average from many independent random draws.

[6] Note that P_{N-1}, the winner's bid, will generally not be observed in ascending auction datasets.

have already dropped out, at prices $P_{k-1}, \ldots, P_0$, respectively. Since the equilibrium bid functions are common knowledge, bidder i can use this information on the identity of the dropout bidders and their dropout prices to infer the private signals $X_{N-k+1}, \ldots, X_N$ observed by these bidders by inverting these bid functions: $X_j = (b_j^{N-j})^{-1}(P_{N-j})$, for $j = N-k+1, \ldots, N$.

The price p at which bidder i should quit the auction during round k, defined as his *bid function* for round k, is the price $b_i^k(X_i) \equiv p$ at which he will have a zero expected profit in round k if all other active bidders simultaneously quit at the same price. In equilibrium, the conditioning event $\mathcal{A}_i^k$, which changes for a given bidder across rounds (therefore the superscript k) consists of (1) bidder i's private signal X_i; (2) the private signals of the bidders who have dropped out before round k, $X_{N-k+1}, \ldots, X_N$, where $X_j = (b_j^{N-j})^{-1}(p_{N-j})$, for $j = N-k+1, \ldots, N$; and (3) bidder i's beliefs that all the other remaining bidders have the same targeted dropout price as he:

$$\begin{aligned} b_1^k(X_1) &= E\left[V_1 \mid X_1, b_j^k(X_j) = b_1^k(X_1), j = 2, \ldots, N-k, \Omega_k\right] \\ b_2^k(X_2) &= E\left[V_2 \mid X_2, b_j^k(X_j) = b_2^k(X_2), j = 1, 3, \ldots, N-k, \Omega_k\right] \\ &\cdots \\ b_{N-k}^k(X_{N-k}) &= E\left[V_{N-k} \mid X_{N-k}, b_j^k(X_j) = b_{N-k}^k(X_{N-k}), j = 1, \ldots, N-k-1, \Omega_k\right] \end{aligned} \tag{11}$$

where $\Omega_k \equiv \left\{ X_j = (b_j^{N-j})^{-1}(P_{N-j}), \text{ for } j = N-k+1, \ldots, N \right\}$. The full set of equilibrium bid functions is analogously described by sets of $N-k$ equations for each round $k = 0, \ldots, N-1$.

In equilibrium, this entire system of equations must hold for any bid p, and the set of signals $\phi_i^k(p) \equiv (b_i^k)^{-1}(p)$, for $i = 1, \ldots, N-k$. If we treat p as a parameter and the inverse bid functions $\phi_i^k)(p)$, $i = 1, \ldots, N-k$ as the unknown variables, we can rewrite (11) as a system of $N-k$ equations in $N-k$ unknowns, analogous to (6):

$$\begin{aligned} p &= E\left[V_1 \mid X_1 = \phi_1^k(p), \ldots, X_{N-k} = \phi_{N-k}^k(p); X_{N-k+1} = \phi_{N-k+1}^{k-1}(p_{k-1}), \ldots, X_N = \phi_N^0(p_0)\right] \\ p &= E\left[V_2 \mid X_1 = \phi_1^k(p), \ldots, X_{N-k} = \phi_{N-k}^k(p); X_{N-k+1} = \phi_{N-k+1}^{k-1}(p_{k-1}), \ldots, X_N = \phi_N^0(p_0)\right] \\ &\cdots \\ p &= E\left[V_{N-k} \mid X_1 = \phi_1^k(p), \ldots, X_{N-k} = \phi_{N-k}^k(p); X_{N-k+1} = \phi_{N-k+1}^{k-1}(p_{k-1}), \ldots, X_N = \phi_N^0(p_0)\right] \end{aligned} \tag{12}$$

where $\phi_i^k(p) \equiv (b_i^k)^{-1}(p)$.

Looping over rounds $k = 0, \ldots, N-2$ and for different values of p, we can solve in pointwise fashion for the set of inverse equilibrium bid functions $(\phi_1^{N-1}(p), \ldots, \phi_i^{N-i}(p), \ldots, \phi_N^0(p))$ which map the observed bids $p_0, \ldots, p_{N-2}$ to the private signals by the relation $X_i = \phi_i^{N-i}(p_{N-i})$.

From a computational point of view, the structure of the round k equilibrium bid functions (12) is particularly attractive since the conditioning events are *points* rather than *sets*, as is the case for asymmetric second price auctions (cf. equations (9)). In the case of the latter, evaluation of conditional expectations would involve multi-dimensional integration, which can be cumbersome as the number of dimensions becomes large.

2.3.5 Consistency conditions in asymmetric ascending auctions

In deriving the joint distribution of bids, $G(p_0, \ldots, p_{N-2})$, in an asymmetric ascending auction models, the researcher conditions on the dropout order observed in the data. In other words, the order of the bids and the identity of their bidders is taken as given in specifying the conditioning events in each round.

However, it is possible that, for some parameter values θ, the bidder signals inferred from the calculated equilibrium bid functions (i.e., the inverse of the set of functions solved in pointwise fashion from systems of equations like those in (12)) imply a dropout order which differs from the observed dropout order. These signals will be inconsistent with the specified form of the equilibrium bid functions, whichtake as given the observed bid ordering.

An example For clarification, we consider a 4-bidder example. If the bid order among 4 bidders is 2,3,1,4, then in deriving the density of bidder 3's bid we condition on this bid order in the sense of assuming that bidder 3 knows bidder 2's signal,[7] and in deriving the density of bidder 1's bid we assume that he has observed the signals of bidders 3 and 2.[8] In specifying bidder 1's equilibrium bidding strategy in round 3, for example, we assume that upon observing bidder 3's exit in the previous round, bidder 1 inverts bidder 3's equilibrium bid function for round 2 at the observed dropout price to obtain X_3, i.e., $X_3 = (b_3^2)^{-1}(p_3;\theta)$. We include θ as an argument here to make explicit the dependence of the bid function on parameters that determine the joint distribution F.

To be more specific, given knowledge of the inverse bid functions $\phi_i^k(p;\theta)$, we can recover bidders' private signals via the relations

$$\begin{aligned} X_1 &= \phi_1^3\,(p_1;\theta) \\ X_2 &= \phi_2^1\,(p_2;\theta) \\ X_3 &= \phi_3^2\,(p_3;\theta) \\ X_4 &= \phi_4^4\,(p_4;\theta)\,. \end{aligned} \tag{13}$$

[7]i.e., $p_3 = b_3^2(X_3) = E[V_1 \mid X_3, X_1 = \phi_1^2(p_3), X_4 = \phi_4^2(p_3); \omega_2 = \{X_2\}]$, which assumes that by round 2, X_2 is already in the public information set ω_2 which bidder 3 conditions upon in forming his bid in that round.

[8]i.e., $p_1 = b_1^3(X_1) = E[V_1 \mid X_1, X_4 = \phi_4^3(p_1); \omega_3 = \{X_2, X_3\}]$, which assumes that by round 3, X_2 and X_3 are already in the public information set ω_1 which bidder 1 conditions upon in forming his bid for that round.

These signals could imply a different dropout order (say, 2,4,1,3) if it were the case that, given θ:

$$\begin{array}{lll} b_2^1(X_2 = \phi_2^1(p_2;\theta);\theta) = \min_{i=1,2,3,4} b_i^1(X_i = \phi_i^1(p_i;\theta);\theta) & \text{i.e., bidder 2 drops out} \\ b_4^2(X_4 = \phi_4^4(p_4;\theta);\theta) = \min_{i=1,3,4} b_i^2(X_i = \phi_i^2(p_i;\theta);\theta) & \text{i.e., bidder 4 drops out} \\ b_1^3(X_1 = \phi_1^3(p_1;\theta);\theta) = \min_{i=1,3} b_i^3(X_i = \phi_i^3(p_i;\theta);\theta) & \text{i.e., bidder 3 wins.} \end{array} \tag{14}$$

where $b_i^k(x) \equiv (\phi_i^k)^{-1}(x)$.

However, our specification of the equilibrium bidding strategies condition on the observed dropout order (2,3,1,4). In other words, bidder 1's observed dropout bid p_1 is modelled as in equation (12):

$$p_1 = b_1^3(X_1) = E[V_i \mid X_1, X_2 = (b_2^1)^{-1}(p_2), X_3 = (b_3^2)^{-1}(p_3), X_4 = (b_4^3)^{-1}(b_1^3(X_1))] \tag{15}$$

If, in fact, given the parameter vector θ, the draw of $X_1, \dots, X_4$ from $F_X(X_1, \dots, X_4 \mid \theta)$ yields the bid ordering (2,4,1,3), then p_1 is modelled as

$$p_1 = b_1^3(X_1) = E[V_i \mid X_1, X_2 = (b_2^1)^{-1}(p_2), X_3 = (b_4^3)^{-1}(b_1^3(X_1)), X_4 = (b_3^2)^{-1}(p_4)] \tag{16}$$

which is clearly inconsistent with equation (15) since $X_4 \neq (b_3^2)^{-1}(p_4)$ and, in equilibrium, $X_3 \neq (b_4^3)^{-1}(b_1^3(X_1))$. ■

For this reason, to ensure consistency with the specified equilibrium bidding strategies, we limit the support of the underlying signals $(X_1, \dots, X_N)$ to regions which — *at the estimated parameter values* θ — would yield the given dropout order. In other words, given parameter values θ[9], we limit the support of $(X_1, \dots, X_N)$ to a region such that

$$b_i^{N-i}(X_i;\theta) = \min_{j=i,\dots,N} b_j^{N-i}(X_j;\theta), \quad \text{for } i = 1, \dots, N. \tag{17}$$

Recall our indexing convention, stated earlier, that bidder i drops out at the end of round $N-i$. For an N-bidder English auction, there will be $N(N-1)/2$ such constraints.[10]

These conditions induce a truncated distribution for the observed bids. Define the set $\mathcal{T}(\theta)$ as the set of draws from $F(X_1, \dots, X_N \mid \theta, \text{observed bid ordering})$ which satisfy the consistency conditions (17), conditional on the values θ.

[9] Unfortunately, as will be discussed below, these truncation conditions depend on θ, the estimated parameters of the F distribution, which violates a regularity condition required for asymptotic normality of the maximum likelihood estimator.

[10] For the log-normal model in Hong and Shum (1997), these constraints can be expressed as a set of inequalities which are linear in the observed dropout prices and nonlinear in the model parameters.

Then the joint density function of all the observed bids is given by:

$$g\left(p_0, \ldots, p_{N-2}\right) = \begin{cases} \dfrac{f\left(\phi_N^0(p_0;\theta), \ldots, \phi_2^{N-2}(p_{N-2};\theta)\right) * J(X_2, \ldots, X_N \mid p_0, \ldots, p_{N-2})}{P(\mathcal{A})} & \text{if } \vec{X} \in \mathcal{T}(\theta) \\ 0 \quad \text{otherwise} & \end{cases} \tag{18}$$

where, as before, J is the Jacobian of the transformation from the private signals to the observed prices.[11]

Note that this consistency problem does not appear in symmetric models, where (cf. Milgrom and Weber (1982) pp. 1104-5) $b_i^j(x) = b_{i'}^j(x)$, for all bidders i, i' and rounds j. The symmetry assumption implies that, for all parameter values θ and across all rounds, the ordering of the signals will always be the same as the ordering of targeted dropout prices. Conditioning on the observed dropout order, then, is enough to ensure that, for all possible values of θ, any draw from $F_X(X_1, \ldots, X_N \mid \theta)$ will satisfy the consistency conditions (17).

For a different reason, consistency problems do not arise in single-round auction models *regardless of bidder asymmetries*. Even though conditioning on the observed bid ordering is not enough to ensure that draws from $F_X(X_1, \ldots, X_N \mid \theta)$ will satisfy the restrictions in (17) when bidder asymmetries are present, *any* ordering of the bids will be consistent with the specified equilibrium bidding strategies in single-round auctions. This is because in single-round auctions, unlike the ascending auction considered above, bidder i never learns the private signal of any other bidder, so that it is possible to observe a given bid independent of the realized bids for any of the other bidders. This is not the case in the ascending auction models in which a bidder's equilibrium bid depends on the realized bids for the bidders who have already dropped out.

3 Estimation strategies

3.1 Minimum distance (least squares) estimation

Once $g(p_1, \ldots, p_N; \theta)$, the joint density (with parameters θ) of the observed bids, has been derived, various estimators of θ are available. The first we consider is minimum distance (method of moments) estimation, which attempts to match sample moments of the observed bids against theoretical moments of the G distribution, computed at each parameter value.

[11] Given the nonlinearity of the consistency conditions, the truncation probability $P(\mathcal{A})$ will likely require simulation methods to compute. See Hong and Shum (1997) for more details.

Even under assumptions of symmetry and independence, the moments of G may not have any analytical solution for first price auction models and maybe difficult to evaluate numerically. The moments of order statistics become even more difficult to calculate if the researcher can only observe winning bid data. Furthermore, for ascending auction incorporating asymmetric and common value assumption, recall that G is a multivariate distribution, the moments of which are multivariate integrals which often do not exist in closed form. Numerical integration techniques are inadequte once the dimension of g (i.e., the number of bidders in the auction) exceeds 4. For these reasons, we suggest adapting the simulated method of moments approach of Laffont, Ossard, and Vuong (1995), in which the moments are approximated using Monte Carlo integration techniques. sdf

We consider a least squares objective function, i.e., an estimator which minimizes the sum of squared deviations between the moments in the data and the theoretical sample moments of the G distribution:

$$(1/T)\sum_{t}\sum_{k=1}^{N_t}(p_k^t - E_g p_k^t)^2 \tag{19}$$

where T is the number of auctions, N_t is the number of bidders in the tth auction, p_k^t is the kth bid in the tth auction, and the expectation is taken with respect to the G distribution, which perhaps does not exist in closed form.

The procedure for simulating $E_g p_k^t$ takes the following steps for draws $s = 1, \ldots, S$:

1. For each parameter vector θ that characterizes the joint distribution F, draw $X_1^s, \ldots, X_N^s$ from the marginal distribution F_X, holding the seed constant for random number generation across different values of θ.

2. Given the parameter value θ, evaluate the bids which correspond to the drawn signals: $p_1^s = b_1(X_1^s; \theta), \ldots, p_N^s = b_N(X_N^s; \theta)$.

3. (For asymmetric ascending auction only) If the bids $p_1^s, \ldots, p_N^s$ satisfy the consistency conditions (17), we retain this draw. Otherwise we discard this draw and repeat the above until we obtain a draw which satisfies these consistency conditions. This is the simplest type of "acceptance/rejection" method for sampling from a conditional distribution.[12]

Given S draws (or S accepted draws, for the case of the asymmetric ascending auction), we ap-

[12]More sophisticated sampling schemes, such as the sequential GHK simulator and Gibbs sampling, are described in Hajivassiliou and McFadden (1998).

proximate the first moment of the bids as:

$$\text{Simulated } E_g p_k^t = (1/S) \sum \sum_s p_k^s, k = 1, \ldots, N_t. \tag{20}$$

Under standard conditions, this nonlinear least squares estimator is consistent and asymptotically normal, as S and N approach ∞.[13]

This approach is applicable to any auction model, provided we can derive the G distribution. Given that we make assumptions about the distribution of the unobserved X's, we throw away information by only using the first moments for purposes of estimation. The distribution of the winning bid in an auction will be asymmetric, even assuming that the private values themselves are drawn from symmetric distributions. Therefore, in situations where only the winning bid is observed (as in Laffont, Ossard, and Vuong (1995)), nonlinear regression which attempts only to match the observed winning bids to the *mean* of the winning bid distribution can be particularly inefficient.[14] Presumably, this problem would be less severe in situations where the researcher observes all of the bids from a given auction.

3.2 Maximum likelihood estimation

Direct maximum likelihood estimation, on the other hand, utilizes all the information embodied in the researcher's distributional assumptions.

However, for several auction models, it turns out that equilibrium behavior of the bidders implies that the support of the observed bids depends parameters of the F distribution, which we are trying to estimate. This violates a regularity condition which ensures consistency and asymptotic normality of the maximum likelihood estimator. Next, we discuss several examples which have arisen in the literature.

[13]Laffont, Ossard, and Vuong (1995) showed that, due to the linearity of their simulator in the draws, the simulated nonlinear least squares estimator is consistent even with a fixed number of simulated draws. However, the asymptotic variance of the estimator must be adjusted to take into account the variance introduced by the finite number of simulate draws. However, for the case of the asymmetric ascending auction with common value components, as in Hong and Shum (1997), the truncation probability also needs to be simulated for each vector of observed bids, and in the case the simulated moment becomes nonlinear in simulation draws. Therefore the simulated nonlinear least square estimator in this case is only consistent when the number of simulated draws increases with the sample size.

[14]Thanks to Samita Sareen for this insight.

3.2.1 First price auctions

For first-price auction models, researchers have uncovered two situations in which equilibrium bidding behavior implies that the support of the data depends on the parameters of the F distribution. Assume that the (common) support of each X_i is $[\underline{x}, \overline{x}]$.

Firstly, Laffont, Ossard, and Vuong (1995) note that in the symmetric IPV first-price auction they consider, the upper bound of the support of the winning bid is

$$E(max(X(1:N-1),p_0) = \int_{p_0}^{\bar{x}} x\,(N-1)\,h\,(x)\,H\,(x)^{I-2}\,dx = \bar{x} - \int_{p_0}^{\bar{x}} H\,(x)^{I-1}\,dx,$$

the unconditional expectation of the maximum of the reservation price and the highest order statistic out of $N-1$ draws from the H distribution.[15] This expectation will be a function of the parameters of the H distribution.

Secondly, Donald and Paarsch (1993) note that for the same auction model in which all the bids are observed, the upper bound of the support for any bid typically depends on the parameters of the H distribution. To see this, consider the equilibrium bid function for this model in equation (1), which is reproduced here:

$$b(X_i) = X_i - \frac{1}{(H(X_i))^{N-1}} \int_{p_0}^{X_i} (H(X_i))^{N-1} dx. \tag{21}$$

Given that $b(X_i)$ is increasing in X_i, the upper bound of the support of any observed bid is $b(\overline{x})$ — the bid that a bidder who observes a signal $\overline{x}$ would submit. This will be a function of the parameters of the H distribution.[16]

While this property violates the regularity conditions on maximum likelihood estimation, Hong (1998) shows that the MLE is still consistent, but no longer $\sqrt{T}$-asymptotically normal. In fact, the MLE is super-consistent, converging at rate T to a mixture of exponential distributions.

Similar problems arise in asymmetric and non-independent first-price auction models, such as that considered by Bajari (1996). The absence of a clear asymptotic theory for the MLE in these multivariate models favors minimum distance or Bayesian estimation techniques.

3.2.2 Open auctions

Under the IPV assumption, no standard regularity conditions are violated in the second-price and ascending auctions models described earlier, because the equilibrium bid function is simply the

[15] Recall that for the symmetric IPV case, $F_X(X_1, \ldots, X_N)$ factors into $H(X_1)H(X_2)\ldots H(X_N)$.

[16] An exception is where $\overline{x} = +\infty$, in which case $\lim_{x\to+\infty} b(x) = +\infty$ and the regularity condition holds.

identity function, and the support of p_i is therefore $[\underline{x}, \overline{x}]$, independently of θ.

However, these problems will crop up again in asymmetric models. As we pointed out earlier, the consistency restrictions (17) impose truncation conditions on the support of the bids observed in an asymmetric ascending auction which depend on θ. Unlike the constraints in first-price auction models discussed in the previous section, these constraints are multivariate (e.g., $l(p_1, \ldots, p_N; \theta) \geq 0$). Very little work has been done on the asymptotics of the MLE in these cases. In particular, although the maximum likelihood estimator is still consistent, its asymptotic distribution is unknown.

4 Conclusions

This paper illustrates in general terms the basic methodology of structural estimation using auction data. It provides a unified view of the common structure underlying structural econometric auction models under various model paradigms. We identify the most crucial steps in building a structural econometric model and discuss the estimation strategies for implementing these models.

The close dependence on a game-theoretic foundation is most the main advantage and disadvantage of structural auction models. While the economic theory provides an efficient framework for econometric estimation and allows for sharp prediction from the estimation results, a structural model is not robust to misspecification and to deviations between the assumptions in theoretical models and the rules of real-world auctions. Exactly the opposite can be said about reduced form approaches.

A compromise between the structural and the reduced form approaches would be to use very general behavioral assumptions — general enough to apply across a number of auction paradigms — in deriving the mapping between bidders' signals and their observed bids, thus retaining the flavor of structural modeling without relying fully on the equilibrium specifications of theoretical auction models. Recent work by Haile (1998) follows such an approach.

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